

***K*-THEORY FOR 2-CATEGORIES**

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ABSTRACT. We establish an equivalence of homotopy theories between symmetric monoidal bicategories and connective spectra. For this, we develop the theory of Γ -objects in 2-categories. In the course of the proof we establish strictification results of independent interest for symmetric monoidal bicategories and for diagrams of 2-categories.

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1. INTRODUCTION

The classifying space functor gives a way of constructing topological spaces from categories. Extra structure on the categorical side often results in extra structure on the corresponding spaces, and categorical coherence theorems give rise to homotopical coherence. This is particularly the case of symmetric monoidal categories. Up to group completion, the classifying space of a symmetric monoidal category is an infinite loop space, as proven independently by Segal [Seg74] (using Γ -spaces) and May [May72] (using operads). The corresponding spectrum is called the K -theory of the symmetric monoidal category. The name comes from the following key example: when the input is the category of finitely generated projective modules over a ring R , the corresponding spectrum is the algebraic K -theory spectrum of the ring whose homotopy groups are then the K -groups of R .

Classically, K -theory developed as a strategy for converting algebraic data to homotopical data. Thomason [Tho95] proved that K -theory establishes an equivalence of homotopy categories between connective spectra and symmetric monoidal categories. Mandell gives an alternate approach in [Man10] combining an equivalence of homotopy categories between Γ -spaces and permutative categories with the Quillen equivalence between connective spectra and Γ -spaces of Bousfield and Friedlander [BF78].

In this work we prove that the K -theory functor defined in [Oso12] induces an equivalence of homotopy theories between symmetric monoidal bicategories and connective spectra. One motivation for this level of generality is that a number of interesting symmetric monoidal structures are naturally 2-categorical. Among these are the 2-category of finite categories, the cobordism bicategory, and bicategories arising from rings, algebras, and bimodules. A K -theory for 2-categories allows us to work directly with this 2-dimensional algebra.

One important general example is that of the bicategory \mathcal{Mod}_R of finitely generated modules over a bimonoidal category R , defined in [BDR04]. By results of [BDRR11] and [Oso12], the bicategorical K -theory of \mathcal{Mod}_R is equivalent to the K -theory of the ring spectrum KR . In the case when R is the topological category of finite dimensional complex vector spaces, we get that the K -theory spectrum of connective complex topological K -theory, $K(ku)$, is equivalent to $K(2Vect_{\mathbb{C}})$, where $2Vect_{\mathbb{C}}$ is the bicategory of 2-vector spaces. Thus, $K(ku)$ is the classifying spectrum for 2-vector bundles (see [BDR04]).

We are also interested in the converse application, namely using K -theory to provide algebraic models for stable homotopical phenomena. This gives another motivation to develop K -theory for 2-categories, namely that 2-dimensional homotopical data is most naturally reflected in 2-dimensional algebra. The first Postnikov invariant of a connective spectrum is readily discernable in the algebra of permutative (1-)categories (see [JO12]), but the most natural algebraic models for n th Postnikov invariants are n -categorical.

In particular, we are interested in the homotopy theory of stable homotopy 2-types and a characterization in terms of the algebra of grouplike symmetric monoidal 2-groupoids. We develop this in a sequel, and explore applications to the Brauer theory of commutative rings.

The primary goal of this paper is to prove the following theorem.

Theorem 1.1. *There are equivalences of homotopy theories*

$$(\Gamma\text{-}2\mathbf{Cat}, \mathcal{S}) \simeq (\mathbf{Perm}2\mathbf{Cat}, \mathcal{S}) \simeq (\mathbf{Perm}\mathbf{Gray}\mathbf{Mon}, \mathcal{S})$$

and therefore induced equivalences of stable homotopy categories.

On the far left, we have Γ -objects in the category of 2-categories and 2-functors, and it is a relatively simple exercise in model category theory (see Proposition 2.16) to show that this is yet another model for the homotopy theory of connective spectra. On the far right, we have permutative Gray-monoids (Definition 3.27) which model symmetric monoidal bicategories categorically. In the middle, we have permutative 2-categories (Definition 3.44). These model symmetric monoidal bicategories homotopically but not categorically, and admit a K -theory functor landing in Γ -2-categories which is simpler and better behaved than the K -theory functor defined on all permutative Gray-monoids. In each case, we have a notion of stable equivalence, denoted \mathcal{S} , and the resulting homotopy theories are constructed using relative categories and complete Segal spaces. Thus we see that Theorem 1.1 gives an equivalence of homotopy theories, and not just homotopy categories, between connective spectra and symmetric monoidal bicategories.

These same techniques can be applied in the 1-categorical case to generalize [Man10, Theorem 1.3] to an equivalence of homotopy theories between Γ -categories and symmetric monoidal categories with their respective stable equivalences.

Outline. In this section we outline the contents of the paper and explain how they combine to prove the main result.

In Section 2 we review some preliminary homotopy theory. In Section 2.1 we recall the basic theory of complete Segal spaces necessary for our work. In Section 2.2 we define Γ -objects in the category of 2-categories and show that these model all connective spectra.

In Section 3 we describe a number of different symmetric monoidal structures on 2-categories or bicategories. We show that every symmetric monoidal bicategory is (categorically) equivalent to a permutative Gray-monoid (Definition 3.27) and every symmetric monoidal pseudofunctor can be replaced, up to a zigzag of strict monoidal equivalences, by a strict monoidal functor between permutative Gray-monoids. We also define permutative 2-categories (Definition 3.44), a stricter notion of primarily homotopical interest. We will eventually show that every permutative Gray-monoid is *weakly* equivalent to a permutative 2-category—see Proposition 6.30 and Theorem 6.44.

Section 4 establishes the groundwork we need for general diagrams of 2-categories. The essential concepts are a notion of lax maps between diagrams and a Grothendieck construction on symmetric monoidal diagrams. The latter is necessary to define the inverse to K -theory, and the former is necessary to define the lax unit for K -theory and its inverse. We end with a construction which takes as input lax maps and produces spans of strict maps. This is the key construction which allows us to replace the lax unit of Section 7.1 with a zigzag of strict equivalences.

In Section 5 we define the inverse to our K -theory functor. It might seem strange to define an inverse to K -theory before defining K -theory itself, but the majority of our methods in this section are entirely standard techniques in enriched category theory or 2-category theory. The results follow from our work in Section 4.

In Section 6 we define the K -theory functor, or more precisely two different K -theory functors, one for permutative Gray-monoids and one for permutative 2-categories. We also study the composite of K -theory followed by its inverse, and show this is naturally weakly equivalent to the identity functor on permutative Gray-monoids.

The final section of the paper, Section 7, studies the composite of the inverse K -theory functor followed by K -theory, and in particular a unit for a putative adjunction between these two functors. Such a unit exists only as a lax map of diagrams, but we show that this lax unit is a stable equivalence and use this to prove our main theorem in two parts.

Theorem 7.25 establishes the right-hand equivalence, and Theorem 7.27 establishes the left-hand equivalence.

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2. PRELIMINARIES

This section will introduce some of the background homotopy theory we will use throughout the paper. Our main result, Theorem 1.1, establishes equivalences between three different homotopy theories, and for this we use the machinery of complete Segal spaces [Rez01]. We give the relevant definitions and some very basic results in Section 2.1. Since our ultimate goal is to show that connective spectra can be modeled categorically, we must choose how to represent these spectra for the comparison. Here we have chosen to use Γ -spaces, together with their class of stable equivalences, as our basic notion of the homotopy theory of connective spectra. Therefore the second goal of this section is to show that we can use 2-categories with their Thomason model structure instead of spaces when thinking about Γ -objects, and we explain this in Section 2.2. We also recall the category Γ itself, together with some related and useful categories, and fix some notation.

2.1. Complete Segal spaces. Complete Segal spaces are bisimplicial sets satisfying certain homotopical properties, and the complete Segal space model structure on bisimplicial sets is constructed as a localization of the Reedy model structure. We will assume some familiarity with these concepts; the interested reader can consult [Rez01, Hir03]. It should be noted that this theory can be easily taken as a black box: Corollary 2.9 suffices on its own for all of our applications.

Definition 2.1. A bisimplicial set X is a *complete Segal space* [Rez01] if

- it is fibrant in the Reedy model structure on bisimplicial sets,
- for each n , the Segal map $X(n) \rightarrow X(1) \times_{X(0)} \cdots \times_{X(0)} X(1)$ is a weak equivalence of simplicial sets, and
- the map $\text{Map}(E, X) \rightarrow \text{Map}(\Delta[1], X) \cong X(1)$, induced by the inclusion of the arrow category into the free living isomorphism (where we take nerves, and then treat the resulting simplicial sets as discrete bisimplicial sets), is a weak equivalence of simplicial sets.

One of the main results of [Rez01] is the following.

Theorem 2.2. *There is a model structure on the category of bisimplicial sets which is a left Bousfield localization of the Reedy model structure such that the fibrant objects are precisely the complete Segal spaces.*

In practice, one constructs complete Segal spaces from model categories, or more generally relative categories.

Definition 2.3. A *relative category* is a pair $(\mathcal{C}, \mathcal{W})$ in which \mathcal{C} is a category and \mathcal{W} is a subcategory of \mathcal{C} containing all of the objects. A *relative functor* $F: (\mathcal{C}, \mathcal{W}) \rightarrow (\mathcal{C}', \mathcal{W}')$ is a functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ such that F restricts to a functor $\mathcal{W} \rightarrow \mathcal{W}'$.

Notation 2.4. Let $(\mathcal{C}, \mathcal{W})$ be a relative category, and let \mathcal{A} be any category. Then $(\mathcal{C}, \mathcal{W})^{\mathcal{A}}$ will denote the category whose objects are functors $\mathcal{A} \rightarrow \mathcal{C}$ and whose morphisms are those natural transformations with components in \mathcal{W} .

We let N denote the usual nerve functor $N: \mathcal{C}\mathcal{at} \rightarrow \mathcal{sSet}$.

Definition 2.5. Let $(\mathcal{C}, \mathcal{W})$ be a relative category. The *classification diagram* of $(\mathcal{C}, \mathcal{W})$ is the bisimplicial set $\mathcal{N}(\mathcal{C}, \mathcal{W})$ given by

$$n \mapsto N((\mathcal{C}, \mathcal{W})^{\Delta[n]}),$$

where as usual $\Delta[n]$ is the category of n composable arrows.

The classification diagram of a relative category is rarely a complete Segal space, one must usually take a fibrant replacement. This can often be done using only a *Reedy* fibrant replacement (see [Rez01, Hir03]), but for an arbitrary relative category one sometimes need to take a fibrant replacement in the complete Segal space model structure.

Definition 2.6. Let $(\mathcal{C}, \mathcal{W})$ be a relative category. The *homotopy theory* of $(\mathcal{C}, \mathcal{W})$ is given by a fibrant replacement of the classification diagram of $(\mathcal{C}, \mathcal{W})$ in the complete Segal space model structure. We write such a fibrant replacement as $R\mathcal{N}(\mathcal{C}, \mathcal{W})$. A relative functor $F: (\mathcal{C}, \mathcal{W}) \rightarrow (\mathcal{C}', \mathcal{W}')$ is an *equivalence of homotopy theories* if the morphism RNF is a weak equivalence in the complete Segal space model structure.

Recall the hammock localization [DK80] of a relative category $(\mathcal{C}, \mathcal{W})$ is a simplicially-enriched category $L^H(\mathcal{C}, \mathcal{W})$. One key property of hammock localization is that the category of components $\pi_0 L^H(\mathcal{C}, \mathcal{W})$ is equivalent to the categorical localization $\mathcal{C}[\mathcal{W}^{-1}]$. A DK-equivalence of simplicially-enriched categories is a simplicially-enriched functor which induces weak equivalences on mapping simplicial sets and for which the induced functor on categories of components is an equivalence of categories.

Proposition 2.7 ([BK12, 1.8]). *A relative functor is an equivalence of homotopy theories if and only if it induces a DK-equivalence on hammock localizations.*

One can actually say something stronger using [Toë05], namely that the classification diagram of a relative category is equivalent to the hammock localization, so that both give the same notion of homotopy category.

The following lemma is an immediate consequence of the complete Segal space model structure being a localization of the Reedy model structure on bisimplicial sets.

Lemma 2.8. *Let $F: (\mathcal{C}, \mathcal{W}) \rightarrow (\mathcal{C}', \mathcal{W}')$ be a relative functor with the property that composition with F induces a weak equivalence of categories*

$$N((\mathcal{C}, \mathcal{W})^{\Delta[n]}) \rightarrow N((\mathcal{C}', \mathcal{W}')^{\Delta[n]})$$

for each n . Then F is an equivalence of homotopy theories.

Corollary 2.9. *Let $F: (\mathcal{C}, \mathcal{W}) \rightarrow (\mathcal{C}', \mathcal{W}')$, $G: (\mathcal{C}', \mathcal{W}') \rightarrow (\mathcal{C}, \mathcal{W})$ be relative functors and let $\alpha: FG \Rightarrow \text{id}$, $\beta: GF \Rightarrow \text{id}$ be natural transformations. If the components of α are all in \mathcal{W}' and the components of β are all in \mathcal{W} , then F, G induce weak equivalences between $R\mathcal{N}(\mathcal{C}, \mathcal{W})$ and $R\mathcal{N}(\mathcal{C}', \mathcal{W}')$ in the complete Segal space model structure. This conclusion remains valid if α, β are replaced with any finite zigzag of transformations between relative functors such that the components of the zigzag replacing α are all in \mathcal{W}' and the components of the zigzag replacing β are all in \mathcal{W} .*

Proof. The transformations α, β induce weak homotopy equivalences between the nerves of $\mathcal{C}, \mathcal{C}'$, and thus of $\mathcal{C}^{\Delta[n]}, \mathcal{C}'^{\Delta[n]}$ as well. The hypotheses on the components of α, β ensure that they restrict to natural transformations when the morphisms of $\mathcal{C}^{\Delta[n]}, \mathcal{C}'^{\Delta[n]}$ are restricted to those maps which are componentwise in $\mathcal{W}, \mathcal{W}'$, respectively. \square

2.2. Γ -2-categories. For a natural number $m \geq 0$ we let \underline{m} be the set with m elements $\{1, \dots, m\}$ where $\underline{0} = \emptyset$. We let be \underline{m}_+ the pointed set $\{0, 1, \dots, m\}$ with basepoint 0.

- \mathcal{N} denotes the category with objects \underline{m} and maps of sets.
- \mathcal{F} denotes the category with objects \underline{m}_+ for $m \geq 0$ and pointed maps.

Notation 2.10. Throughout, we let $2\mathit{Cat}$ denote the (1-)category of 2-categories and strict 2-functors. We let $2\mathit{Cat}_2$ denote the 2-category of 2-categories, 2-functors, and 2-natural transformations.

Definition 2.11. For any category \mathcal{C} with a terminal object $*$, a Γ -object X in \mathcal{C} is a functor $X: \mathcal{F} \rightarrow \mathcal{C}$ such that $X(\underline{0}_+) = *$. The term *reduced* is also used for this condition on $X(\underline{0}_+)$. We will denote by $\Gamma\text{-}\mathcal{C}$ the category of Γ -objects and natural transformations. We will denote by $\mathcal{F}\text{-}\mathcal{C}$ the category of all functors $\mathcal{F} \rightarrow \mathcal{C}$, and then $\Gamma\text{-}\mathcal{C}$ is the full subcategory of $\mathcal{F}\text{-}\mathcal{C}$ consisting of reduced \mathcal{F} -diagrams in \mathcal{C} .

Our K -theory will produce Γ -2-categories, where Segal's category Γ is the opposite of \mathcal{F} .

There are several possible definitions of nerve of a 2-category. However there are natural weak equivalences connecting any pair of such definitions (see [CCG10]) so it is not necessary to choose a particular classifying space functor in order to make the next definition.

Definition 2.12. Let \mathcal{A}, \mathcal{B} be 2-categories, and $F: \mathcal{A} \rightarrow \mathcal{B}$ be a 2-functor between them.

- i. F is an *equivalence* of 2-categories if each object $b \in \mathcal{B}$ is equivalent to Fa for some $a \in \mathcal{A}$, and if each functor $\mathcal{A}(a, a') \rightarrow \mathcal{B}(Fa, Fa')$ is an equivalence of categories.
- ii. F is a *weak equivalence* if the induced map on nerves $NF: N\mathcal{A} \rightarrow N\mathcal{B}$ is a weak equivalence of simplicial sets.

Remark 2.13. For the definition of equivalence of 2-categories above, it is useful to recall that two objects b, b' in a 2-category \mathcal{B} are equivalent, or internally equivalent, if there exist 1-cells $f: b \rightarrow b', g: b' \rightarrow b$ and isomorphism 2-cells $\alpha: fg \cong \text{id}_{b'}, \beta: gf \cong \text{id}_b$. Similarly, a 1-cell $f: b \rightarrow b'$ is an internal equivalence in \mathcal{B} if g, α, β exist as above. One can then check that the definition of equivalence of 2-categories given above amounts to what is usually called a biequivalence. It is not the case, though, that if a 2-functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is a biequivalence then there is necessarily a 2-functor $G: \mathcal{B} \rightarrow \mathcal{A}$ which is a biequivalence and a weak inverse for F . In general, one can only produce such a G as a pseudofunctor.

There are two model structures on the category of 2-categories and 2-functors. The first has equivalences as its class of weak equivalences, and was established by Lack [Lac02b, Lac04]. The second, called the Thomason model structure, has the weak equivalences (as given in the definition above) for its class of weak equivalences (in the model categorical sense), and was established by [WHPT07] with corrections by [AM13]. By standard arguments, every equivalence is a weak equivalence, so any functor which inverts weak equivalences also inverts equivalences. Further, any functor inverting weak equivalences must identify 2-functors with a 2-natural (or lax, or oplax) transformation between them by [CCG10]. In Section 4.2 we will express this fact using a functorial path object so that we can apply the same technique to diagrams of 2-categories.

Notation 2.14. Let $\text{ho}\mathcal{F}\text{-}2\mathit{Cat}$ denote the category obtained from $\mathcal{F}\text{-}2\mathit{Cat}$ by inverting the maps which are levelwise weak equivalences, and $\text{ho}\Gamma\text{-}2\mathit{Cat}$ denote the category obtained from $\Gamma\text{-}2\mathit{Cat}$ by inverting the maps which are levelwise weak equivalences.

Remark 2.15. Since the Thomason model structure on $2\mathcal{C}at$ is cofibrantly generated, we can produce the homotopy category $ho\mathcal{F}\text{-}2\mathcal{C}at$ using the generalized Reedy model structure in which weak equivalences are levelwise. As noted in the passing remark of [BF78, p. 95], this restricts to a generalized Reedy model structure on $\Gamma\text{-}2\mathcal{C}at$ because for each $n \geq 1$ the category of Σ_n -objects in $2\mathcal{C}at$ has the projective model structure.

The Quillen equivalence between $s\mathcal{S}et$ and $2\mathcal{C}at$ of [WHPT07, AM13] yields a Quillen equivalence between the respective generalized Reedy model structures on $\mathcal{F}\text{-}s\mathcal{S}et$ and $\mathcal{F}\text{-}2\mathcal{C}at$. Since both functors of the adjunction between $s\mathcal{S}et$ and $2\mathcal{C}at$ preserve the terminal object, the corresponding adjunction between $\mathcal{F}\text{-}s\mathcal{S}et$ and $\mathcal{F}\text{-}2\mathcal{C}at$ restricts to give the following.

Proposition 2.16. *There is a Quillen equivalence between $\Gamma\text{-}s\mathcal{S}et$ and $\Gamma\text{-}2\mathcal{C}at$, both with the model structures described above.*

Thus we can transport definitions directly from the context of Γ -spaces (here seen as Γ -simplicial sets) to Γ -2-categories. Proposition 2.16 implies, in particular, that $ho\Gamma\text{-}2\mathcal{C}at$ is locally small.

Definition 2.17. Let X be a Γ -2-category.

- i. Let $i_k : \underline{n}_+ \rightarrow \underline{1}_+$ denote the unique map sending only k to the non-basepoint element in $\underline{1}_+$. Then X is *special* if the map $X(\underline{n}_+) \rightarrow X(\underline{1}_+)^n$ induced by all the i_k is a weak equivalence of 2-categories.
- ii. A special Γ -2-category X is *very special* when $\pi_0 X(\underline{1}_+)$ is a group under the operation

$$\pi_0 X(\underline{1}_+) \times \pi_0 X(\underline{1}_+) \cong \pi_0 X(\underline{2}_+) \rightarrow \pi_0 X(\underline{1}_+)$$

where the isomorphism is induced by the weak equivalence $X(\underline{2}_+) \rightarrow X(\underline{1}_+)^2$.

Definition 2.18. A map $f : X \rightarrow Y$ in $\Gamma\text{-}2\mathcal{C}at$ is a *stable equivalence* if the function

$$f^* : ho\Gamma\text{-}2\mathcal{C}at(Y, Z) \rightarrow ho\Gamma\text{-}2\mathcal{C}at(X, Z)$$

is an isomorphism for every very special Γ -2-category Z .

Notation 2.19. Let $Ho\Gamma\text{-}2\mathcal{C}at$ denote the category obtained from $\Gamma\text{-}2\mathcal{C}at$ by inverting the stable equivalences, and $Ho\Gamma\text{-}s\mathcal{S}et$ denote the category obtained from $\Gamma\text{-}s\mathcal{S}et$ by inverting the stable equivalences.

Definition 2.20. Let \mathcal{W} denote the collection of weak equivalences in $\Gamma\text{-}2\mathcal{C}at$ and \mathcal{S} the collection of stable equivalences. Then $(\Gamma\text{-}2\mathcal{C}at, \mathcal{W})$ and $(\Gamma\text{-}2\mathcal{C}at, \mathcal{S})$ are the respective relative categories. Similarly, $(\Gamma\text{-}s\mathcal{S}et, \mathcal{W})$ is the relative category of Γ -simplicial sets and levelwise weak equivalences, and $(\Gamma\text{-}s\mathcal{S}et, \mathcal{S})$ is the relative category of Γ -simplicial sets and stable equivalences.

Note that $Ho\Gamma\text{-}s\mathcal{S}et$ is (equivalent to) the full subcategory of the stable homotopy category whose objects are connective spectra [BF78]. The remarks above prove the following theorem.

Theorem 2.21. *Applying the adjunction between $2\mathcal{C}at$ and $s\mathcal{S}et$ levelwise, we obtain an adjunction between $\Gamma\text{-}2\mathcal{C}at$ and $\Gamma\text{-}s\mathcal{S}et$. This induces an equivalence of homotopy theories*

$$(\Gamma\text{-}2\mathcal{C}at, \mathcal{S}) \xrightarrow{\simeq} (\Gamma\text{-}s\mathcal{S}et, \mathcal{S})$$

and therefore an equivalence of stable homotopy categories

$$Ho\Gamma\text{-}2\mathcal{C}at \xrightarrow{\simeq} Ho\Gamma\text{-}s\mathcal{S}et.$$

3. SYMMETRIC MONOIDAL 2-CATEGORIES

In this section we give basic results on symmetric monoidal structures for 2-categories and bicategories. We first outline the general theory of symmetric monoidal bicategories in Section 3.1 and then describe the stricter notions of permutative Gray-monoid and permutative 2-category in Sections 3.3 and 3.4, respectively. Both descriptions make use of the Gray tensor product, which we recall in Section 3.2.

3.1. Symmetric monoidal bicategories. While our main objects of study, permutative Gray-monoids and permutative 2-categories, are 2-categories (see Definition 3.27 and Definition 3.44), it is important for us to use some of the more general theory of symmetric monoidal bicategories. Since we use only some main results of this theory and not the particular details of many definitions, we only sketch the notions of symmetric monoidal bicategory together with functors and transformations between them. Good references for these details include [SP11, McC00, Lac10, CG14a]. In this section, functor always means pseudofunctor, transformation always means pseudonatural transformation (which we will only indicate via components), and equivalence always means pseudonatural adjoint equivalence.

Definition 3.1 (Sketch, see [SP11, Defn. 2.3] or [CG14a]). A *symmetric monoidal bicategory* consists of

- a bicategory \mathcal{B} ,
- a tensor product functor $\mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$,
- a unit object $e \in \text{ob } \mathcal{B}$,
- an associativity equivalence $(xy)z \simeq x(yz)$,
- unit equivalences $xe \simeq x \simeq ex$
- four invertible modifications between composites of the unit and associativity equivalences
- a braid equivalence $\beta : xy \simeq yx$,
- two invertible modifications (denoted $R_{--| -}, R_{-| --}$) which correspond to two instances of the third Reidemeister move,
- and an invertible modification (the syllepsis, v , which has components indexed by pairs of objects) between $\beta \circ \beta$ and the identity

satisfying three axioms for the monoidal structure, four axioms for the braided structure, two axioms for the sylleptic structure, and one final axiom for the symmetric structure.

Remark 3.2. In the theory of monoidal categories, there are notions of braided and symmetric structures, but a sylleptic structure is a new intermediate structure that only appears in the 2-categorical context. The syllepsis axioms compare $v_{xy,z}$ with the composite of $v_{x,z}$ and $v_{y,z}$ (and similarly in the second variable), while the symmetry axiom ensures that there is a unique isomorphism between β^3 and β . Moreover, the coherence theorem for symmetric monoidal bicategories in [GO13] states that the axioms in this definition imply there is a unique structural isomorphism between any compositions of the associativity, unit, and braiding equivalences which represent the same permutation of objects.

Definition 3.3 (Sketch, see [SP11, Defn. 2.5]). A *symmetric monoidal pseudofunctor* $F : \mathcal{B} \rightarrow \mathcal{C}$ consists of

- a functor $F : \mathcal{B} \rightarrow \mathcal{C}$,
- a unit equivalence $F(e_{\mathcal{B}}) \simeq e_{\mathcal{C}}$,
- an equivalence for the tensor product $FxFy \simeq F(xy)$,

- three invertible modifications between composites of the unit and tensor product equivalences, and
- an invertible modification U comparing the braidings in \mathcal{B} and \mathcal{C}

satisfying two axioms for the monoidal structure, two axioms for the braided structure, and one axiom for the symmetric (and hence subsuming the sylleptic) structure.

Definition 3.4 (Sketch, see [SP11, Defn. 2.7]). A *symmetric monoidal transformation* $\alpha : F \rightarrow G$ consists of

- a transformation $\alpha : F \rightarrow G$ and
- two invertible modifications concerning the interaction between α and the unit objects on the one hand and the tensor products on the other

satisfying two axioms for the monoidal structure and one axiom for the symmetric structure (and hence subsuming the braided and sylleptic structures).

The following is verified in [SP11]. Note that we have not defined symmetric monoidal modifications as we will not have any reason to use them in any of our constructions.

Lemma 3.5. *There is a tricategory of symmetric monoidal bicategories, symmetric monoidal pseudofunctors, symmetric monoidal transformations, and symmetric monoidal modifications.*

We will need to know when symmetric monoidal pseudofunctors or transformations are invertible in the appropriate sense.

Definition 3.6. A *symmetric monoidal biequivalence* $F : \mathcal{B} \rightarrow \mathcal{C}$ is a symmetric monoidal pseudofunctor such that the underlying functor F is a biequivalence of bicategories.

Definition 3.7. A *symmetric monoidal equivalence* $\alpha : F \rightarrow G$ between symmetric monoidal pseudofunctors is a symmetric monoidal transformation $\alpha : F \rightarrow G$ such that the underlying transformation α is an equivalence. This is logically equivalent to the condition that each component 1-cell $\alpha_b : Fb \rightarrow Gb$ is an equivalence 1-cell in \mathcal{C} .

The results of [Gur12] can be used to easily prove the following lemma, although the first part is also verified by elementary means in [SP11].

Lemma 3.8. *Let $F, G : X \rightarrow Y$ be symmetric monoidal pseudofunctors, and $\alpha : F \rightarrow G$ a symmetric monoidal transformation between them.*

- $F : X \rightarrow Y$ is a symmetric monoidal biequivalence as above if and only if it is an internal biequivalence in the tricategory SMBicat .
- $\alpha : F \rightarrow G$ is a symmetric monoidal equivalence if and only if it is an internal equivalence in the bicategory $\text{SMBicat}(\mathcal{B}, \mathcal{C})$.

We now come to the definition and results from [SP11] that are most important for our construction of K -theory later.

Definition 3.9 ([SP11, Def 2.28]). A symmetric monoidal bicategory \mathcal{B} is a *quasi-strict symmetric monoidal 2-category* if

- the underlying monoidal bicategory of \mathcal{B} is a Gray-monoid (see Definition 3.16),
- the braided structure is strict in the sense of Crans [Cra98], and
- the following three additional axioms hold.

QS1 The modifications $R_{--| -}, R_{-| --}, v$ are all identities.

QS2 The naturality 2-cells

$$\begin{array}{ccc}
 ab & \xrightarrow{f1} & a'b \\
 \beta \downarrow & \cong \beta_{f1} & \downarrow \beta \\
 ba & \xrightarrow{1f} & ba'
 \end{array}
 \quad
 \begin{array}{ccc}
 ab & \xrightarrow{1g} & ab' \\
 \beta \downarrow & \cong \beta_{1g} & \downarrow \beta \\
 ba & \xrightarrow{g1} & b'a
 \end{array}$$

for the pseudonatural transformation β are identities.

QS3 The 2-cells $\Sigma_{\beta,g}, \Sigma_{f,\beta}$ (see Definition 3.15) are the identity for any 1-cells f,g .

Remark 3.10.

- i. The first axiom above implies that the tensor product is given by a cubical functor, and that this operation is strictly unital and associative. It is the functoriality isomorphism for the tensor product as a cubical functor that gives rise to the 2-cells Σ that appear in (QS3). We will discuss cubical functors and their properties in the next section.
- ii. Our version of (QS3) is not exactly as it is presented in [SP11], but it is equivalent using the cubical functor axioms. We find this version more amenable to our later work.

There is also a stricter notion of morphism that will be of interest to us.

Definition 3.11. A *strict functor* $F : \mathcal{B} \rightarrow \mathcal{C}$ between symmetric monoidal bicategories is a strict functor of the underlying bicategories that preserves all of the structure strictly, and for which all of the constraints are either the identity (if this makes sense) or unique coherence isomorphisms from \mathcal{C} .

Remark 3.12. There is a monad on the category of 2-globular sets whose algebras are symmetric monoidal bicategories. Strict functors can then be identified with the morphisms in the Eilenberg-Moore category for this monad, and in particular symmetric monoidal bicategories with strict functors form a category. This point of view is crucial to the methods employed in [SP11].

The following strictification theorem of [SP11] enables us to restrict attention to quasi-strict symmetric monoidal 2-categories. Its proof relies heavily on the coherence theorem for symmetric monoidal bicategories in [GO13].

Theorem 3.13 ([SP11, Thm. 2.97]). *Let \mathcal{B} be a symmetric monoidal bicategory.*

- i. *There are two endofunctors, $\mathcal{B} \mapsto \mathcal{B}^c$ and $\mathcal{B} \mapsto \mathcal{B}^{qst}$, of the category of symmetric monoidal bicategories and strict functors between them. Any symmetric monoidal bicategory of the form \mathcal{B}^{qst} is a quasi-strict symmetric monoidal 2-category.*
- ii. *There are natural transformations $(-)^c \Rightarrow \text{id}, (-)^c \Rightarrow (-)^{qst}$. When evaluated at a symmetric monoidal bicategory \mathcal{B} , these give natural strict biequivalences*

$$\mathcal{B} \leftarrow \mathcal{B}^c \rightarrow \mathcal{B}^{qst}.$$

- iii. *For a symmetric monoidal pseudofunctor $F : \mathcal{B} \rightarrow \mathcal{C}$, there are strict functors $F^c : \mathcal{B}^c \rightarrow \mathcal{C}^c, F^{qst} : \mathcal{B}^{qst} \rightarrow \mathcal{C}^{qst}$ such that the right hand square below commutes and the left hand square commutes up to a symmetric monoidal equivalence.*

$$\begin{array}{ccccc}
 \mathcal{B} & \xleftarrow{\quad} & \mathcal{B}^c & \xrightarrow{\quad} & \mathcal{B}^{qst} \\
 F \downarrow & \simeq & \downarrow F^c & & \downarrow F^{qst} \\
 \mathcal{C} & \xleftarrow{\quad} & \mathcal{C}^c & \xrightarrow{\quad} & \mathcal{C}^{qst}
 \end{array}$$

We will need a strengthening of the above result in the special case that we begin with quasi-strict symmetric monoidal 2-categories instead of the more general symmetric monoidal bicategories.

Theorem 3.14. *Let \mathcal{B} be a quasi-strict symmetric monoidal 2-category.*

- i. *There is a strict functor $v: \mathcal{B}^{qst} \rightarrow \mathcal{B}$ such that*

$$\begin{array}{ccc} \mathcal{B}^c & \xrightarrow{\quad} & \mathcal{B}^{qst} \\ & \searrow & \swarrow v \\ & \mathcal{B} & \end{array}$$

commutes, where the unlabeled morphisms are those from Theorem 3.13. In particular, v is a strict symmetric monoidal biequivalence.

- ii. *For a symmetric monoidal pseudofunctor $F: \mathcal{B} \rightarrow \mathcal{C}$ with \mathcal{B}, \mathcal{C} with both quasi-strict, the square below commutes up to a symmetric monoidal equivalence.*

$$\begin{array}{ccc} \mathcal{B}^{qst} & \xrightarrow{\quad v \quad} & \mathcal{B} \\ F^{qst} \downarrow & \simeq & \downarrow F \\ \mathcal{C}^{qst} & \xrightarrow{\quad v \quad} & \mathcal{C} \end{array}$$

Proof. The first statement follows from the general theory of computads developed in [SP11] and the calculus of mates [KS74]. We conclude that v is a strict symmetric monoidal biequivalence since the other two maps in the triangle are, and biequivalences satisfy the 2-out-of-3 property. The second claim follows directly from the third part of Theorem 3.13. \square

In Theorem 3.42 we repackage the definition of quasistrict symmetric monoidal 2-category using the Gray tensor product. Before doing so, we give a basic review of the Gray tensor product in the next section.

3.2. 2-categories and the Gray tensor product. Let 2Cat denote the category of strict 2-categories and strict 2-functors between them. We will often be concerned with a monoidal structure on 2Cat which is not the cartesian structure, but instead uses the tensor product defined below. For further reference, see [Gra74, GPS95, Gur13].

Definition 3.15. Let \mathcal{A}, \mathcal{B} be 2-categories. The Gray tensor product of \mathcal{A} and \mathcal{B} , written $\mathcal{A} \otimes \mathcal{B}$ is the 2-category given by

- 0-cells consisting of pairs $a \otimes b$ with a an object of \mathcal{A} and b an object of \mathcal{B} ;
- 1-cells generated by basic 1-cells of the form $f \otimes 1: a \otimes b \rightarrow a' \otimes b$ for $f: a \rightarrow a'$ in \mathcal{A} and $1 \otimes g: a \otimes b \rightarrow a \otimes b'$ for $g: b \rightarrow b'$ in \mathcal{B} , subject to the relations
 - $(f \otimes 1)(f' \otimes 1) = (ff') \otimes 1$,
 - $(1 \otimes g)(1 \otimes g') = 1 \otimes (gg')$
and with identity 1-cell $\text{id} \otimes 1 = 1 \otimes \text{id}$; and
- 2-cells generated by basic 2-cells of the form $\alpha \otimes 1$, $1 \otimes \delta$, and $\Sigma_{f,g}: (f \otimes 1)(1 \otimes g) \cong (1 \otimes g)(f \otimes 1)$ subject to the relations
 - $(\alpha \otimes 1) \cdot (a' \otimes 1) = (\alpha \cdot a') \otimes 1$,
 - $(1 \otimes \delta) \cdot (1 \otimes \delta') = 1 \otimes (\delta \cdot \delta')$,

where \cdot can be taken as either vertical or horizontal composition of 2-cells, together with the axioms below for 2-cells of the form $\Sigma_{f,g}$.

The basic properties of the Gray tensor product are summarized below.

- i. The unit object for this monoidal structure is the terminal 2-category, just as it is for the cartesian monoidal structure.
- ii. This monoidal structure is closed, with corresponding internal hom functor denoted $[\mathcal{B}, \mathcal{C}]$. The 0-cells of $[\mathcal{B}, \mathcal{C}]$ are the 2-functors $F : \mathcal{B} \rightarrow \mathcal{C}$, the 1-cells $\alpha : F \rightarrow G$ are the pseudonatural transformations from F to G , and the 2-cells $\Gamma : \alpha \Rightarrow \beta$ are the modifications. In particular, each functor $-\otimes \mathcal{B}$ has a right adjoint $[\mathcal{B}, -]$ and therefore preserves all colimits.
- iii. This monoidal structure is symmetric, with $\tau : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ sending $a \otimes b$ to $b \otimes a$, $f \otimes 1$ to $1 \otimes f$, $1 \otimes g$ to $g \otimes 1$, similarly on 2-cells, and then extended on arbitrary 1-

and 2-cells in these generators by 2-functoriality. On the cell $\Sigma_{f,g}$, we then have

$$\tau(\Sigma_{f,g}) = \Sigma_{g,f}^{-1}.$$

iv. This monoidal structure also equips $2\mathcal{C}\mathcal{A}\mathcal{T}$ with the structure of a monoidal model category using the canonical model structure developed in [Lac02b], although we will not make use of this structure here.

For the reader new to the Gray tensor product, perhaps the most helpful feature to remark upon is that the 1- and 2-cells are generated by certain basic cells subject to relations. In particular, an arbitrary 1-cell of $\mathcal{A} \otimes \mathcal{B}$ has the form

$$(f_1 \otimes 1)(1 \otimes g_1) \cdots (f_n \otimes 1)(1 \otimes g_n)$$

for some natural number n and some 1-cells f_i of \mathcal{A} and g_i of \mathcal{B} .

As with all closed, symmetric monoidal categories, we not only get an isomorphism of hom-sets

$$2\mathcal{C}\mathcal{A}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong 2\mathcal{C}\mathcal{A}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]),$$

but an isomorphism of hom-objects, in this case 2-categories, of the form

$$[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \cong [\mathcal{A}, [\mathcal{B}, \mathcal{C}]].$$

In particular one can transport pseudonatural transformations and modifications between the tensor product and hom-2-category side.

Definition 3.16. A *Gray-monoid* is a monoid object in $2\mathcal{C}\mathcal{A}\mathcal{T}$ with the Gray tensor product, \otimes . This consists of a 2-category \mathcal{C} , a 2-functor

$$\oplus : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C},$$

an object e of \mathcal{C} satisfying associativity and unit axioms.

The Gray tensor product has another universal property relating it to the notion of cubical functor.

Definition 3.17. A pseudofunctor $F : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \rightarrow \mathcal{B}$ is *cubical* if the following condition holds:

if $(f_1, f_2, \dots, f_n), (g_1, g_2, \dots, g_n)$ is a composable pair of morphisms in the 2-category $\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$ such that for all $i > j$, either g_i or f_j is an identity map, then the comparison 2-cell

$$\phi : F(f_1, f_2, \dots, f_n) \circ F(g_1, g_2, \dots, g_n) \Rightarrow F((f_1, f_2, \dots, f_n) \circ (g_1, g_2, \dots, g_n))$$

is an identity.

When $n = 1$, it is easy to check that a cubical functor is the same thing as a 2-functor. When $n = 2$, we have the following characterization, a proof of which appears in either [GPS95] or [Gur13].

Proposition 3.18. A cubical functor $F : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathcal{B}$ determines, and is uniquely determined by

i. For each object $a_1 \in ob\mathcal{A}_1$, a strict 2-functor $F_{a_1} : \mathcal{A}_2 \rightarrow \mathcal{B}$, and for each object $a_2 \in ob\mathcal{A}_2$, a strict 2-functor $F_{a_2} : \mathcal{A}_1 \rightarrow \mathcal{B}$, such that for each pair of objects a_1, a_2 in $\mathcal{A}_1, \mathcal{A}_2$, respectively, the equation

$$F_{a_1}(a_2) = F_{a_2}(a_1) := F(a_1, a_2)$$

holds;

ii. For each pair of 1-cells $f_1 : a_1 \rightarrow a'_1$, $f_2 : a_2 \rightarrow a'_2$ in $\mathcal{A}_1, \mathcal{A}_2$, respectively, a 2-cell isomorphism

$$\begin{array}{ccc} F(a_1, a_2) & \xrightarrow{F_{a_1}(f_2)} & F(a_1, a'_2) \\ F_{a_2}(f_1) \downarrow & \swarrow \Sigma_{f_1, f_2} \cong & \downarrow F_{a'_2}(f_1) \\ F(a'_1, a_2) & \xrightarrow{F_{a'_1}(f_2)} & F(a'_1, a'_2) \end{array}$$

which is an identity 2-cell if either f_1 or f_2 is an identity 1-cell;

subject to the following 3 axioms for all diagrams of the form

$$\begin{array}{ccccc} & & (f_1, f_2) & & \\ (a_1, a_2) & \xrightarrow{\downarrow \downarrow (a_1, a_2)} & (a'_1, a'_2) & \xrightarrow{(h_1, h_2)} & (a''_1, a''_2) \\ & \xrightarrow{(g_1, g_2)} & & & \end{array}$$

in $\mathcal{A}_1 \times \mathcal{A}_2$.

$$\begin{array}{ccc} & F_{a_1}(f_2) & \\ & \swarrow \downarrow F_{a_1} \alpha_2 \quad \searrow & \\ F(a_1, a_2) & \xrightarrow{F_{a_1}(g_2)} & F(a_1, a'_2) \\ F_{a_2}(g_1) \left(\begin{array}{c} F_{a_2} \alpha_1 \\ \swarrow \quad \searrow \end{array} \right) & F_{a_2}(f_1) \quad \Downarrow \Sigma & \downarrow F_{a'_2}(f_1) \\ F(a'_1, a_2) & \xrightarrow{F_{a'_1}(g_2)} & F(a'_1, a'_2) \end{array} = \begin{array}{ccc} F(a_1, a_2) & \xrightarrow{F_{a_1}(f_2)} & F(a_1, a'_2) \\ \downarrow F_{a_2}(g_1) & & \downarrow \Sigma F_{a'_2}(g_1) \left(\begin{array}{c} F_{a'_2} \alpha_1 \\ \swarrow \quad \searrow \end{array} \right) F_{a'_2}(f_1) \\ F(a'_1, a_2) & \xrightarrow{F_{a'_1}(f_2)} & F(a'_1, a'_2) \\ \downarrow \downarrow F_{a'_1} \alpha_2 & \nearrow & \\ F_{a'_1}(g_2) & & \end{array}$$

$$\begin{array}{ccc} F(a_1, a_2) & \xrightarrow{F_{a_1}(f_2)} & F(a_1, a'_2) \\ F_{a_2}(f_1) \downarrow & \Downarrow \Sigma & \downarrow F_{a'_2}(f_1) \\ F(a'_1, a_2) & \xrightarrow{F_{a'_1}(f_2)} & F(a'_1, a'_2) \\ F_{a_2}(h_1) \downarrow & \Downarrow \Sigma & \downarrow F_{a'_2}(h_1) \\ F(a''_1, a_2) & \xrightarrow{F_{a''_1}(f_2)} & F(a''_1, a'_2) \end{array} = \begin{array}{ccc} F(a_1, a_2) & \xrightarrow{F_{a_1}(f_2)} & F(a_1, a'_2) \\ \downarrow F_{a_2}(h_1 f_1) & & \downarrow \Sigma F_{a'_2}(h_1 f_1) \\ F(a''_1, a_2) & \xrightarrow{F_{a''_1}(f_2)} & F(a''_1, a'_2) \end{array}$$

$$\begin{array}{ccccc}
F(a_1, a_2) & \xrightarrow{F_{a_1}(f_2)} & F(a_1, a'_2) & \xrightarrow{F_{a_1}(h_2)} & F(a_1, a''_2) \\
\downarrow F_{a_2}(f_1) & \Downarrow \Sigma & \downarrow F_{a'_2}(f_1) & \Downarrow \Sigma & \downarrow F_{a''_2}(f_1) \\
F(a'_1, a_2) & \xrightarrow{F_{a'_1}(f_2)} & F(a'_1, a'_2) & \xrightarrow{F_{a'_1}(h_2)} & F(a'_1, a''_2) \\
& & \parallel & & \\
F(a_1, a_2) & & F(a_1, h_2 f_2) & & F(a_1, a''_2) \\
\downarrow F_{a_2}(f_1) & & \Downarrow \Sigma & & \downarrow F_{a''_2}(f_1) \\
F(a'_1, a_2) & & F_{a'_1}(h_2 f_2) & & F(a'_1, a''_2)
\end{array}$$

The following proposition is easy to check, and appears in [Gur13].

Proposition 3.19. *There is a multicategory 2Cat_c whose objects are 2-categories and for which the set*

$$2\text{Cat}_c(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n; \mathcal{B})$$

consists of the cubical functors $\mathcal{A}_1 \times \dots \times \mathcal{A}_n \rightarrow \mathcal{B}$.

Theorem 3.20. *Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be 2-categories. There is a cubical functor*

$$c : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B},$$

natural in \mathcal{A} and \mathcal{B} , such that composition with c induces an isomorphism

$$2\text{Cat}_c(\mathcal{A}, \mathcal{B}; \mathcal{C}) \cong 2\text{Cat}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}).$$

Sketch proof, see [Gur13]. We define c using Proposition 3.18. We define the 2-functor c_a by

$$\begin{aligned}
c_a(b) &= (a, b) \\
c_a(f) &= (1_a, f) \\
c_a(\alpha) &= (1_{1_a}, \alpha);
\end{aligned}$$

the 2-functor c_b is defined similarly. The 2-cell isomorphism $\Sigma_{f,g}$ is the same $\Sigma_{f,g}$ that is part of the data for $\mathcal{A} \otimes \mathcal{B}$.

To prove that this cubical functor has the claimed universal property, assume that $F : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ is a cubical functor. We define a strict 2-functor $\bar{F} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ by the following formulas.

$$\begin{aligned}
\bar{F}(a, b) &= F(a, b) \\
\bar{F}(f, 1) &= F_b(f) \\
\bar{F}(1, g) &= F_a(g) \\
\bar{F}(\alpha, 1) &= F_b(\alpha) \\
\bar{F}(1, \beta) &= F_a(\beta) \\
\bar{F}(\Sigma_{f,g}^{\mathcal{A} \otimes \mathcal{B}}) &= \Sigma_{f,g}^F
\end{aligned}$$

□

For our discussion of permutative Gray-monoids later, we will need the notion of an opcubical functor.

Definition 3.21. A pseudofunctor $F : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \rightarrow \mathcal{B}$ is *opcubical* if the following condition holds:

if $(f_1, f_2, \dots, f_n), (g_1, g_2, \dots, g_n)$ is a composable pair of morphisms in the 2-category $\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$ such that for all $i < j$, either g_i or f_j is an identity map, then the comparison 2-cell

$$\phi : F(f_1, f_2, \dots, f_n) \circ F(g_1, g_2, \dots, g_n) \Rightarrow F((f_1, f_2, \dots, f_n) \circ (g_1, g_2, \dots, g_n))$$

is an identity.

As the difference between a cubical functor and an opcubical one is merely a matter of ordering, we immediately get the following lemma.

Lemma 3.22. *There is a bijection, natural in all variables, between the set of cubical functors $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ and the set of opcubical functors $\mathcal{B} \times \mathcal{A} \rightarrow \mathcal{C}$. In particular, if the pseudofunctor $F : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ is cubical, then*

$$\mathcal{B} \times \mathcal{A} \cong \mathcal{A} \times \mathcal{B} \xrightarrow{F} \mathcal{C}$$

is *opcubical*.

If $F : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ is cubical, we can produce another opcubical functor F^* , this time with source $\mathcal{A} \times \mathcal{B}$, by defining

$$F^*(f, g) = F(f, 1)F(1, g)$$

and replacing the necessary structure 2-cells with their inverses. This process was introduced in [GPS95] and is called nudging. One can check that $F^* \cong F$ as pseudofunctors in the 2-category of 2-categories, pseudofunctors, and icons [Lac10], and therefore we obtain an isomorphism between the set of cubical functors $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ and the set of opcubical functors $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$. This procedure, together with the lemma above, is one method for giving the symmetry isomorphism for the Gray tensor product. Moreover, one can show that the map from the Gray tensor product to the cartesian one $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A} \times \mathcal{B}$ induced by the identity 2-functor $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \times \mathcal{B}$ viewed as a cubical functor is a map of symmetric monoidal structures. It is not hard to check, though, that the universal cubical functor $c : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}$ is only symmetric up to an invertible icon.

Lemma 3.23. *Let $F : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ be a cubical functor, and let $\bar{F} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ be the associated 2-functor. Then the cubical functor associated to the 2-functor*

$$\mathcal{B} \otimes \mathcal{A} \xrightarrow{\tau_\otimes} \mathcal{A} \otimes \mathcal{B} \xrightarrow{\bar{F}} \mathcal{C}$$

is $F^* \tau_\times$, where F^* is the opcubical functor associated to F as defined above.

Proof. We must show that $\overline{F^* \tau_\times} = \bar{F} \tau_\otimes$ by checking that the two agree on generating cells, including cells of the form $\Sigma_{f,g}$. On generating cells arising directly from one of the copies of \mathcal{A} , this is obvious. For the $\Sigma_{f,g}$'s, we have

$$\begin{aligned} \bar{F} \tau_\otimes(\Sigma_{f,g}) &= \bar{F}((\Sigma_{g,f})^{-1}) \\ &= (\bar{F}(\Sigma_{g,f}))^{-1} \\ &= (\Sigma_{g,f}^F)^{-1} \\ &= \overline{F^* \tau_\times}(\Sigma_{f,g}) \end{aligned}$$

by the definition of τ_\otimes , 2-functors of the form \bar{G} for a cubical functor G , and the opcubical functor F^* . \square

Notation 3.24. Given 1-cells f, g in a Gray-monoid, we let $f \oplus g$ denote $\overline{\oplus}(f, g)$, where $\overline{\oplus}$ is the cubical functor associated to \oplus . Concretely, $f \oplus g = (\text{id} \oplus g) \circ (f \oplus \text{id})$. Similarly, for 1-cells f_1, \dots, f_q we let $\oplus_i f_i = \overline{\oplus}(f_1, \dots, f_q)$, where now $\overline{\oplus}$ is the cubical functor associated to the iterated sum $\oplus: \mathcal{C}^{\otimes q} \rightarrow \mathcal{C}$.

The next two subsections will study two different strict notions of what one might consider a symmetric monoidal 2-category. The first is that of a permutative Gray-monoid which we will show is equivalent to the notion of quasi-strict symmetric monoidal 2-category introduced in [SP11]. We believe that this repackaging of the definition sheds conceptual light, and helps to motivate the second definition, that of a permutative 2-category. While permutative 2-categories are not equivalent to permutative Gray-monoids in the categorical (or bicategorical, or even tricategorical) sense, we will later show that they do have the same homotopy theory.

3.3. Permutative Gray-monoids. We begin with a reminder of the definition of a permutative category.

Definition 3.25. A *permutative category* C consists of a strict monoidal category (C, \oplus, e) together with a natural isomorphism,

$$\begin{array}{ccc} C \times C & \xrightarrow{\tau} & C \times C \\ \oplus \swarrow \downarrow \beta \searrow \oplus & & \\ C & & \end{array}$$

where $\tau: C \times C \rightarrow C \times C$ is the symmetry isomorphism in Cat , such that the following axioms hold for all objects x, y, z of C .

- $\beta_{y,x} \beta_{x,y} = \text{id}_{x \oplus y}$
- $\beta_{e,x} = \text{id}_x = \beta_{x,e}$
- $\beta_{x,y \oplus z} = (y \oplus \beta_{x,z}) \circ (\beta_{x,y} \oplus z)$

Remark 3.26. It is relatively easy to check that this definition is logically equivalent to the definition of a symmetric monoidal category with underlying monoidal structure strict.

Each of the axioms for a permutative category can be expressed in a purely diagrammatic form and thus studied in other contexts. We do so now in the context of 2-categories using the Gray tensor product.

Definition 3.27. A *permutative Gray-monoid* \mathcal{C} consists of a Gray-monoid (\mathcal{C}, \oplus, e) together with a 2-natural isomorphism,

$$\begin{array}{ccc} \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\tau} & \mathcal{C} \otimes \mathcal{C} \\ \oplus \swarrow \downarrow \beta \searrow \oplus & & \\ \mathcal{C} & & \end{array}$$

where $\tau: \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ is the symmetry isomorphism in 2Cat for the Gray tensor product, such that the following axioms hold.

- The following pasting diagram is equal to the identity 2-natural transformation for the 2-functor \oplus .

$$\begin{array}{ccc} & 1 & \\ & \swarrow \tau \quad \searrow \tau & \\ \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\tau} & \mathcal{C} \otimes \mathcal{C} \xrightarrow{\tau} \mathcal{C} \otimes \mathcal{C} \\ \oplus \swarrow \downarrow \beta \quad \uparrow \oplus & & \downarrow \beta \quad \searrow \oplus \\ \mathcal{C} & & \end{array}$$

- The following pasting diagram is equal to the identity 2-natural transformation for the canonical isomorphism $1 \otimes \mathcal{C} \cong \mathcal{C}$.

$$\begin{array}{ccccc}
 1 \otimes \mathcal{C} & \xrightarrow{e \otimes \text{id}} & \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\tau} & \mathcal{C} \otimes \mathcal{C} \\
 & \searrow \cong & \downarrow \oplus & \swarrow \oplus & \\
 & & \mathcal{C} & &
 \end{array}$$

- The following equality of pasting diagrams holds where we have abbreviated the tensor product to concatenation when labeling 1- or 2-cells.

$$\begin{array}{ccc}
 \mathcal{C}^{\otimes 3} & \xrightarrow{\tau \text{id}} & \mathcal{C}^{\otimes 3} \xrightarrow{\text{id} \tau} \mathcal{C}^{\otimes 3} \xrightarrow{\oplus \text{id}} \mathcal{C}^{\otimes 2} \\
 \downarrow \oplus & \searrow \text{id} \oplus & \downarrow \oplus \\
 & = &
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{C}^{\otimes 3} & \xrightarrow{\tau \text{id}} & \mathcal{C}^{\otimes 3} \xrightarrow{\text{id} \tau} \mathcal{C}^{\otimes 3} \xrightarrow{\oplus \text{id}} \mathcal{C}^{\otimes 2} \\
 \downarrow \oplus & \searrow \text{id} \oplus & \downarrow \oplus \\
 & = &
 \end{array}$$

Remark 3.28. Although the definition of permutative Gray-monoid is analogous to the definition of permutative category, there is an important difference. Permutative categories can be described as the algebras for a 2-monad on \mathbf{Cat} , but no such description can be made for permutative Gray-monoids. This is because a 2-monad for permutative Gray-monoids would have as its underlying 2-category that of 2-categories, 2-functors, and 2-natural transformations. But there is no way to extend the Gray tensor product as a functor of categories $\otimes : 2\mathbf{Cat} \times 2\mathbf{Cat} \rightarrow 2\mathbf{Cat}$ to a 2-functor of 2-categories with the same objects and 1-cells but 2-natural transformations as 2-cells: one can easily verify that there is no way to make a Gray tensor product of a pair of 2-natural transformations itself into a 2-natural transformation, it is only possible to produce a pseudonatural one. Therefore such a 2-monad does not exist. Nevertheless, we borrow heavily from the strategies in 2-dimensional algebra in dealing with permutative Gray-monoids, using such notions as strict and (op)lax functors.

Remark 3.29. The definition of permutative Gray-monoid makes no mention of permutations of more than 3 objects, so the reader might wonder about the existence and uniqueness of something deserving to be called $\beta \oplus \beta : x \oplus y \oplus z \oplus w \cong y \oplus x \oplus w \oplus z$. A priori, there are two such 1-cells, $(\text{id} \oplus \beta) \circ (\beta \oplus \text{id})$ and $(\beta \oplus \text{id}) \circ (\text{id} \oplus \beta)$. However in Proposition 3.41 we will see that $\Sigma_{\beta, \beta}$ is the identity, so there is a unique 1-cell isomorphism for any permutation of objects.

Definition 3.30. A *strict functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ of permutative Gray-monoids is a 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of the underlying 2-categories satisfying the following conditions.

- $F(e_{\mathcal{C}}) = e_{\mathcal{D}}$, so that F strictly preserves the unit object.
- The diagram

$$\begin{array}{ccc}
 \mathcal{C} \otimes \mathcal{C} & \xrightarrow{F \otimes F} & \mathcal{D} \otimes \mathcal{D} \\
 \downarrow \oplus_{\mathcal{C}} & & \downarrow \oplus_{\mathcal{D}} \\
 \mathcal{C} & \xrightarrow{F} & \mathcal{D}
 \end{array}$$

commutes, so that F strictly preserves the sum.

- The equation

$$\beta^{\mathcal{D}} * (F \otimes F) = F * \beta^{\mathcal{C}}$$

holds, so that F strictly preserves the symmetry. This equation is equivalent to requiring that

$$\beta_{Fx,Fy}^{\mathcal{D}} = F(\beta_{x,y}^{\mathcal{C}})$$

as 1-cells from $Fx \oplus Fy = F(x \oplus y)$ to $Fy \oplus Fx = F(y \oplus x)$.

Proposition 3.31. *There is a category PermGrayMon of permutative Gray-monoids and strict functors between them. The underlying 2-category functor $\text{PermGrayMon} \rightarrow 2\text{Cat}$ is monadic in the usual, 1-categorical sense.*

Definition 3.32. Let \mathcal{C}, \mathcal{D} be a pair of permutative Gray-monoids. A strict functor of permutative Gray-monoids $F: \mathcal{C} \rightarrow \mathcal{D}$ is a *weak equivalence* if the underlying 2-functor is a weak equivalence of 2-categories. We let $(\text{PermGrayMon}, \mathcal{W})$ denote the relative category of permutative Gray-monoids with weak equivalences,

We also have the notion of lax functor.

Definition 3.33. A *lax functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ of permutative Gray-monoids consists of

- a 2-functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between the underlying 2-categories,
- a 1-cell $\theta_0: e_{\mathcal{D}} \rightarrow F(e_{\mathcal{C}})$, and
- a 2-natural transformation

$$\begin{array}{ccc} \mathcal{C} \otimes \mathcal{C} & \xrightarrow{F \otimes F} & \mathcal{D} \otimes \mathcal{D} \\ \oplus_{\mathcal{C}} \downarrow & \Downarrow \theta & \downarrow \oplus_{\mathcal{D}} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

subject to the requirement that the following diagrams commute for all objects $x, y, z \in \mathcal{C}$.

$$\begin{array}{ccc} e \oplus Fx & \xrightarrow{\theta_0 \oplus 1} & Fe \oplus Fx \\ & \searrow \theta & \downarrow \\ & F(e \oplus x) & \\ & \parallel & \\ & Fx & \end{array} \quad \begin{array}{ccc} Fx \oplus e & \xrightarrow{1 \oplus \theta_0} & Fx \oplus Fe \\ & \searrow \theta & \downarrow \\ & F(x \oplus e) & \\ & \parallel & \\ & Fx & \end{array}$$

$$\begin{array}{ccc} Fx \oplus Fy \oplus Fz & \xrightarrow{\theta \oplus 1} & F(x \oplus y) \oplus Fz \\ \downarrow 1 \oplus \theta & & \downarrow \theta \\ Fx \oplus F(y \oplus z) & \xrightarrow{\theta} & F(x \oplus y \oplus z) \end{array}$$

$$\begin{array}{ccc} Fx \oplus Fy & \xrightarrow{\theta} & F(x \oplus y) \\ \beta \downarrow & & \downarrow F(\beta) \\ Fy \oplus Fx & \xrightarrow{\theta} & F(y \oplus x) \end{array}$$

Just as one can compose lax monoidal functors between monoidal categories, it is possible to compose lax functors between permutative Gray-monoids. If (F, θ_0, θ) is a lax functor $\mathcal{C} \rightarrow \mathcal{D}$ and (G, ψ_0, ψ) is a lax functor $\mathcal{D} \rightarrow \mathcal{A}$, then the composite GF is given the structure of a lax functor with 1-cell

$$e \xrightarrow{\psi_0} Ge \xrightarrow{G\theta_0} GFe$$

and 2-natural transformation with components

$$GFx \oplus GFy \xrightarrow{\psi} G(Fx \oplus Fy) \xrightarrow{G\theta} GF(x \oplus y).$$

There are two additional variants one might consider.

Definition 3.34. A *pseudofunctor* $F : \mathcal{C} \rightarrow \mathcal{D}$ between permutative Gray-monoids is a lax functor in which both the structure 1-cell for the unit object and the 2-natural transformation are isomorphisms. An *oplax functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of

- a 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between the underlying 2-categories,
- a 1-cell $\theta_0 : F(e_{\mathcal{C}}) \rightarrow e_{\mathcal{D}}$, and
- a 2-natural transformation θ with components $F(x \oplus y) \rightarrow Fx \oplus Fy$

subject to axioms such as those in Definition 3.33 with all arrows reversed.

It is clear that pseudofunctors are closed under composition, and one can define the composition of oplax functors in much the same way as it is defined for lax ones.

Remark 3.35. A pseudofunctor in this sense is not the weakest possible notion, as it has an underlying 2-functor.

Definition 3.36. A *normal functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ of permutative Gray-monoids is a functor (lax, oplax, or pseudo) for which θ_0 is the identity.

Since the underlying morphism of a functor (of any kind) between permutative Gray-monoids is itself a 2-functors, the composite of two normal functors of the same kind will be another normal functor. Note also that every strict functor is already normal.

Proposition 3.37. *There are categories of permutative Gray-monoids with lax functors, oplax functors, and pseudofunctors; we denote these with the subscripts l , op , and ps respectively. We also have categories whose maps are the normal variants of each, denoted with respective subscripts nl , nop , and nps . We have canonical inclusions*

$$\begin{array}{ccccc} & & \mathbf{PermGrayMon}_{nl} & \longrightarrow & \mathbf{PermGrayMon}_l \\ & \nearrow & & & \searrow \\ \mathbf{PermGrayMon} & \longrightarrow & \mathbf{PermGrayMon}_{nps} & \rightarrow & \mathbf{PermGrayMon}_{ps} \\ & \searrow & & & \swarrow \\ & & \mathbf{PermGrayMon}_{nop} & \longrightarrow & \mathbf{PermGrayMon}_{op} \end{array}$$

which commute with the forgetful functors to 2Cat .

We return now to a discussion of plain 2-categories without any monoidal structure in order to prove the equivalence between quasi-strict symmetric monoidal 2-categories and permutative Gray-monoids.

Definition 3.38. Let \mathcal{A} and \mathcal{B} be 2-categories and let $F, G : \mathcal{A} \rightarrow \mathcal{B}$ be pseudofunctors. A pseudonatural transformation $\alpha : F \Rightarrow G$ is a *strict transformation* if for each 1-cell $f : a \rightarrow b$, the 2-cell isomorphism

$$\alpha_f : Gf * \alpha_a \cong \alpha_b * Ff$$

is the identity 2-cell.

Remark 3.39. Note that when F, G in the definition above are strict 2-functors rather than pseudofunctors, strict naturality is what is usually called 2-naturality in the 2-categorical literature [KS74]. It will become clear below why we have introduced this additional layer of terminology.

Lemma 3.40. *Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be 2-categories and let $F, G: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ be a pair of pseudofunctors.*

- i. *Assume that F, G are cubical, and $\overline{F}, \overline{G}: \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ are the associated 2-functors. Then there is an isomorphism between the set of pseudonatural transformations $\alpha: F \Rightarrow G$ and the set of pseudonatural transformations $\overline{\alpha}: \overline{F} \Rightarrow \overline{G}$. Furthermore, the following conditions are equivalent:*
 - $\overline{\alpha}$ *is a 2-natural transformation,*
 - α *is a strict transformation, and*
 - *the components $\alpha_{f,1}, \alpha_{1,g}$ are the identity for all 1-cells f in \mathcal{A} and g in \mathcal{B} .*
- ii. *Assume that F, G are opcubical, and $\overline{F}^*, \overline{G}^*: \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ are the associated 2-functors. Then there is an isomorphism between the set of pseudonatural transformations $\alpha: F \Rightarrow G$ and the set of pseudonatural transformations $\overline{\alpha}: \overline{F}^* \Rightarrow \overline{G}^*$. Furthermore, the following conditions are equivalent:*
 - $\overline{\alpha}$ *is a 2-natural transformation,*
 - α *is a strict transformation, and*
 - *the components $\alpha_{f,1}, \alpha_{1,g}$ are the identity for all 1-cells f in \mathcal{A} and g in \mathcal{B} .*
- iii. *Assume that F is cubical and G is opcubical, and $\overline{F}, \overline{G}^*: \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ are the associated 2-functors. Then there is an isomorphism between the set of pseudonatural transformation $\alpha: F \Rightarrow G$ and the set of pseudonatural transformations $\overline{\alpha}: \overline{F} \Rightarrow \overline{G}^*$. Furthermore, the following conditions are equivalent:*
 - $\overline{\alpha}$ *is a 2-natural transformation and*
 - *the components $\alpha_{f,1}, \alpha_{1,g}$ are the identity for all 1-cells f in \mathcal{A} and g in \mathcal{B} .*

Proof. For the first two variants above, define $\overline{\alpha}_{a \otimes b} = \alpha_{a,b}$ for the 0-cell components and $\overline{\alpha}_{f \otimes \text{id}} = \alpha_{f,\text{id}}$, $\overline{\alpha}_{\text{id} \otimes g} = \alpha_{\text{id},g}$ for the 1-cell components on generators. The 1-cell components for a general 1-cell are forced by the axioms, and it is easy to check the pseudonaturality axioms as well as the equivalent conditions stated above. The same definitions work for the third variant, only note that $\alpha_{f,g}$ is not forced to be the identity even if both $\alpha_{f,1}, \alpha_{1,g}$ are since it must satisfy the axiom shown below where the single unlabeled isomorphism is from the pseudofunctoriality of G and is not necessarily the identity as G is only opcubical and not cubical.

$$\begin{array}{ccc}
 \begin{array}{c}
 F(a,b) \xrightarrow{\alpha_{a,b}} G(a,b) \\
 \swarrow F(f,\text{id}) \quad \searrow G(f,\text{id}) \\
 F(f,g) = F(a',b) \xrightarrow{\alpha_{a',b}} G(a',b) \\
 \downarrow F(\text{id},g) \quad \swarrow G(\text{id},g) \\
 F(a',b') \xrightarrow{\alpha_{a',b'}} G(a',b')
 \end{array}
 & \quad &
 \begin{array}{c}
 F(a,b) \xrightarrow{\alpha_{a,b}} G(a,b) \\
 \downarrow \alpha_{f,g} \quad \downarrow G(f,g) \quad \searrow G(f,\text{id}) \\
 F(f,g) = F(a',b') \xrightarrow{\alpha_{a',b'}} G(a',b') \quad \cong G(a',b) \\
 \downarrow G(\text{id},g) \quad \downarrow G(\text{id},g) \\
 G(a',b') \xrightarrow{\alpha_{a',b'}} G(a',b')
 \end{array}
 \end{array}$$

□

Proposition 3.41. *Axioms (QS1) and (QS2) imply (QS3) in the presence of the others.*

Proof. We must show that $\Sigma_{f,\beta}$ and $\Sigma_{\beta,g}$ are identities. We do this for $\Sigma_{\beta,g}$ below; the other case is similar. By the modification axiom for R , we have the indicated equality of

pasting diagrams:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 a \oplus b \oplus c & \xrightarrow{\text{id} \oplus g} & a \oplus b \oplus c' \\
 \beta \oplus \text{id} \downarrow & \cong & \beta \oplus \text{id} \downarrow \\
 b \oplus a \oplus c & \xrightarrow{\text{id} \oplus g} & b \oplus a \oplus c' = \beta \\
 \text{id} \oplus \beta \downarrow & = & \text{id} \oplus \beta \downarrow \\
 b \oplus c \oplus a & \xrightarrow{\text{id} \oplus g \oplus \text{id}} & b \oplus c' \oplus a
 \end{array} & = & \begin{array}{ccc}
 a \oplus b \oplus c & \xrightarrow{\text{id} \oplus g} & a \oplus b \oplus c' \\
 \beta \oplus \text{id} \downarrow & & \beta \oplus \text{id} \downarrow \\
 b \oplus a \oplus c & = \beta & = \beta \\
 \text{id} \oplus \beta \downarrow & & \text{id} \oplus \beta \downarrow \\
 b \oplus c \oplus a & \xrightarrow{\text{id} \oplus g \oplus \text{id}} & b \oplus c' \oplus a
 \end{array}
 \end{array}$$

The two triangular regions marked with equal signs are identities by (QS1), and the two squares marked with equal signs are identities by (QS2). Thus the invertible 2-cell in the remaining square is an identity after whiskering by $\text{id} \oplus \beta$. But since $\text{id} \oplus \beta$ is itself an isomorphism 1-cell, this statement is true before whiskering, thus verifying (QS3). \square

Theorem 3.42. *There is an isomorphism of categories $\text{PermGrayMon} \cong \text{qsSM2Cat}$.*

Proof. Each of the categories above consists of 2-categories equipped with additional structure, together with 2-functors preserving all of that structure. Thus to construct the desired isomorphism, we merely have to show that to give the data for a permutative Gray-monoid structure is the same as to give the data for a quasi-strict structure, and similarly that such data satisfies the axioms for one structure if and only if it satisfies the axioms for the other. This will produce a bijection on objects, and moreover immediately imply a bijection on morphisms.

Recall from Definition 3.9 that a quasi-strict symmetric monoidal 2-category consists of

- a Gray-monoid \mathcal{C} , with *cubical* multiplication $+$: $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and unit object e , with
- a strict braided structure in the sense of Crans [Cra98], the braiding of which we shall call b , such that
- the modifications $R_{-|--}, R_{--|-}, v$ are identities and
- the naturality cells $b_{f,\text{id}}, b_{\text{id},g}$ are identities.

By Proposition 3.41, we have omitted axiom (QS3) as it is redundant, and we have given the multiplication as a cubical functor as that is how it appears in the source and target of b . A permutative Gray-monoid, on the other hand, consists of

- a Gray-monoid \mathcal{C} , with multiplication given by a 2-functor $\oplus: \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$, with
- a 2-natural transformation β such that
- $\beta^2 = 1$,
- β is the identity when either object is the unit object, and
- $\beta_{x,yz}$ is given as a composite of $\beta_{x,y}$ and $\beta_{x,z}$.

In the presence of the rest of the quasi-strict structure, the strict braided structure reduces to the single axiom that $b_{e,x}$ is the identity for any object x . Now the braiding b is a pseudonatural transformation from an opcubical functor (by Lemma 3.22) to a cubical one, but it satisfies the conditions in the third variant listed in Lemma 3.40 so it induces a 2-natural transformation

$$\bar{b}: \overline{(+ \circ \tau_x)^*} \longrightarrow \overline{+}$$

where we have written the cubical functor giving the multiplication as $+$ and the associated 2-functor as $\overline{+}$. Since τ_x is strict, it is easy to check that

$$(+ \circ \tau_x)^* = +^* \circ \tau_x,$$

and by Lemma 3.23 the associated 2-functor is then $\bar{\tau} \circ \tau_\otimes$. Thus we see that the pseudonatural transformation b in the definition of a quasi-strict symmetric monoidal 2-category, subject to the conditions in (QS2), corresponds to the 2-natural transformation β in the definition of a permutative Gray-monoid. The only remaining difference between the two definitions is that the equation $R_{--| -} = 1$ is not explicitly required in the definition of a permutative Gray-monoid, but requiring both $v = 1$ and $R_{-| - -} = 1$ (the third and fifth axioms) shows that $\beta_{xy,z}$ is the appropriate composite of $\beta_{x,z}$ and $\beta_{y,z}$ by noting that the composite

$$(\beta_{x,z} \otimes z) \circ (x \otimes \beta_{y,z}) \circ \beta_{z,xy}$$

is the identity and using the invertibility of β . \square

Remark 3.43. The note [Bar14] gives a slightly different repackaging of quasi-strict symmetric monoidal 2-categories, together with a graphical calculus for 2-cells. Bartlett's focus is on constructing symmetric monoidal bicategories using generators and relations, so such a calculus is crucial for his application, while our focus is more theoretical.

3.4. Permutative 2-categories. We now come to the second strict notion of symmetric monoidal bicategory that we will introduce. This notion is not equivalent to those studied thus far in the categorical sense, but it does have many nice properties such as being described by an operad (see Proposition 3.47).

Definition 3.44. A *permutative 2-category* \mathcal{C} consists of a monoid (\mathcal{C}, \oplus, e) in $(2\text{Cat}, \times)$, together with a 2-natural isomorphism,

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{\tau} & \mathcal{C} \times \mathcal{C} \\ \downarrow \beta & & \downarrow \oplus \\ \mathcal{C} & & \mathcal{C} \end{array}$$

where $\tau: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ is the symmetry isomorphism in 2Cat for the cartesian product, such that the same axioms hold as for permutative Gray-monoids once all the instances of \otimes are replaced with \times .

From a purely categorical point of view, permutative 2-categories are very special creatures. It is not true that every symmetric monoidal bicategory is symmetric monoidal biequivalent to a permutative 2-category (see [SP11, Example 2.30]). On the other hand, we will see that this is a natural structure to consider homotopically: every symmetric monoidal bicategory is weakly equivalent (i.e., homotopy equivalent after passing to nerves) to a permutative 2-category. We prove this in Proposition 6.28 by proving that every permutative Gray-monoid is weakly equivalent to a permutative 2-category. In Theorem 6.44 we prove that this establishes an equivalence of homotopy theories.

Proposition 3.45. *Let $(\mathcal{C}, \oplus, \beta)$ be a permutative Gray-monoid. Then the composite*

$$\mathcal{C} \times \mathcal{C} \xrightarrow{c} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\oplus} \mathcal{C}$$

*of the universal cubical functor c (see Theorem 3.20) with \oplus , together with $\beta * 1_c$, give \mathcal{C} the structure of a permutative 2-category if and only if $\oplus \circ c$ is a 2-functor.*

Proof. Since the axioms are the same, all that remains is to show that $\beta * 1_c$ is 2-natural. By Lemma 3.40, β is 2-natural so $\beta * 1_c$ is a strict transformation from $\oplus c$ to $\oplus \tau_\otimes c$. Let $+: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ be the cubical functor associated to \oplus , so that $\oplus c = +$. We know that the cubical functor associated to $\oplus \tau_\otimes$ is $+^* \circ \tau_\times$, so

$$\beta * 1_c: + \longrightarrow +^* \circ \tau_\times$$

is a strict transformation. But by assumption, $+ = \oplus c$ is a 2-functor, so $+^* = +$ is a 2-functor and strict naturality of $\beta * 1_c$ is then 2-naturality. \square

We now turn to the 2-monadic aspects of the theory of permutative 2-categories.

Definition 3.46. Let 2Cat_2 denote the 2-category consisting of (small) 2-categories, 2-functors, and 2-natural transformations.

Let $E\Sigma_n$ be the translation category of Σ_n , viewed as a discrete 2-category. The $E\Sigma_n$ give a symmetric operad in $(2\text{Cat}, \times)$ and thus a monad S on 2Cat . It is straightforward to check that this is actually a 2-monad on the 2-category 2Cat_2 . This is merely the Cat -enrichment of the operadic approach to permutative categories, and we therefore leave the proof of the next proposition to the reader. Note the contrast with Remark 3.28.

Proposition 3.47. *Permutative 2-categories are precisely the S -algebras in 2Cat_2 .*

Using the 2-monad structure on S , we can make the following definitions. We refer the reader to [BKP89] or [Lac02a] for the general definitions.

Definition 3.48. A *strict functor* between permutative 2-categories is a strict S -algebra morphism. A *pseudo, lax, oplax, or normal functor* is, respectively, a pseudo, lax, oplax, or normal S -algebra morphism.

Theorem 3.49. *Permutative 2-categories, with any choice of morphism above, form a full subcategory of permutative Gray-monoids with the corresponding morphism type.*

Proof. We describe the case of lax morphisms; the other cases are similar. A lax S -algebra morphism $h: \mathcal{C} \rightarrow \mathcal{D}$ consists of [BKP89]

- a 2-functor $h: \mathcal{C} \rightarrow \mathcal{D}$ between the underlying 2-categories and
- a 2-natural transformation

$$\begin{array}{ccc} S\mathcal{C} & \xrightarrow{Sh} & S\mathcal{D} \\ \downarrow & \Downarrow \nu & \downarrow \\ \mathcal{C} & \xrightarrow{h} & \mathcal{D} \end{array}$$

where the two vertical arrows are the S -algebra structures on \mathcal{C}, \mathcal{D} ,

satisfying two axioms. Using the structure of the operad S and following the calculations in [CG14b], we see that this amounts to the following data and axioms:

- a 2-functor $h: \mathcal{C} \rightarrow \mathcal{D}$ between the underlying 2-categories,
- a 1-cell $\nu_0: e_{\mathcal{D}} \rightarrow h(e_{\mathcal{C}})$, and
- a 2-natural transformation

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{h \times h} & \mathcal{D} \times \mathcal{D} \\ \downarrow \oplus_{\mathcal{C}} & \Downarrow \nu & \downarrow \oplus_{\mathcal{D}} \\ \mathcal{C} & \xrightarrow{h} & \mathcal{D} \end{array}$$

subject to the requirement that the following diagrams commute for all objects $x, y, z \in \mathcal{C}$.

$$\begin{array}{ccc}
 e \oplus h(x) & \xrightarrow{v_0 \oplus 1} & h(e) \oplus h(x) \\
 \searrow & \downarrow v & \searrow \\
 & h(e \oplus x) & \\
 & \parallel & \\
 & h(x) &
 \end{array}
 \quad
 \begin{array}{ccc}
 h(x) \oplus e & \xrightarrow{1 \oplus v_0} & h(x) \oplus h(e) \\
 \searrow & \downarrow v & \searrow \\
 & h(x \oplus e) & \\
 & \parallel & \\
 & h(x) &
 \end{array}$$

$$\begin{array}{ccc}
 h(x) \oplus h(y) \oplus h(z) & \xrightarrow{v \oplus 1} & h(x \oplus y) \oplus h(z) \\
 \downarrow 1 \oplus v & & \downarrow v \\
 h(x) \oplus h(y \oplus z) & \xrightarrow[v]{} & h(x \oplus y \oplus z)
 \end{array}$$

$$\begin{array}{ccc}
 h(x) \oplus h(y) & \xrightarrow{v} & h(x \oplus y) \\
 \beta \downarrow & & \downarrow h(\beta) \\
 h(y) \oplus h(x) & \xrightarrow[v]{} & h(y \oplus x)
 \end{array}$$

Thus we see that there is a bijection between lax S -algebra morphisms $\mathcal{C} \rightarrow \mathcal{D}$ and lax functors $\mathcal{C} \rightarrow \mathcal{D}$ in the sense of the cartesian analogue of Definition 3.33 in which all instances of \otimes are replaced with instances of \times .

To complete the proof, we must show that if (h, θ, θ_0) is a lax functor between permutative 2-categories viewed as permutative Gray-monoids via Proposition 3.45, then $(h, \theta * 1_c, \theta_0)$ is a lax functor using this cartesian analogue of Definition 3.33. Just as in Proposition 3.45, the axioms are the same so this reduces to showing that $\theta * 1_c$ is 2-natural. This fact is easily verified using the same methods as those which show that $\beta * 1_c$ is 2-natural in the proof of Proposition 3.45. \square

Definition 3.50. We let Perm2Cat denote the category of permutative 2-categories and strict functors.

We have a relative category of permutative 2-categories and weak equivalences defined as follows.

Definition 3.51. Let \mathcal{C}, \mathcal{D} be a pair of permutative 2-categories. A strict functor of permutative 2-categories $F: \mathcal{C} \rightarrow \mathcal{D}$ is a *weak equivalence* if the underlying 2-functor is a weak equivalence of 2-categories. We let $(\text{Perm2Cat}, \mathcal{W})$ denote the relative category of permutative 2-categories with weak equivalences.

4. DIAGRAMS OF 2-CATEGORIES

In this section we give the basic theory we will need for diagrams of 2-categories. In addition to diagrams indexed on \mathcal{F} , we will use diagrams in 2Cat indexed on the category \mathcal{A} of Definition 5.7. Therefore we present the theory for a general diagram category \mathcal{D} .

Definition 4.1. Given a category \mathcal{D} , let $\mathcal{D}\text{-}2\text{Cat}$ denote the category of functors and natural transformations from \mathcal{D} to 2Cat .

Recall that Γ -objects in 2Cat are defined (via Definition 2.11) to be those functors $X: \mathcal{F} \rightarrow 2\text{Cat}$ with $X(\underline{0}_+) = *$.

We will need to consider a category whose objects are \mathcal{D} -2-categories but whose morphisms are lax in a particular sense—we describe this in Section 4.1 and describe transformations between them in Section 4.2. In Section 4.3 we describe a Grothendieck construction for \mathcal{D} -2-categories and lax maps between them. In Section 4.4 we extend this theory to symmetric monoidal diagrams for later use in Section 5.3. Finally, Section 4.5 gives a general construction which allows us to replace lax maps with spans of strict ones.

4.1. Lax maps of diagrams. Recall that (small) 2-categories, 2-functors, and 2-natural transformations naturally organize themselves into a 2-category we denote 2Cat_2 .

Definition 4.2. Let X, Y be \mathcal{D} -2-categories, viewed as 2-functors $X, Y : \mathcal{D} \rightarrow 2\text{Cat}_2$ with \mathcal{D} seen as a locally discrete 2-category, i.e., one with only identity 2-cells. Then a \mathcal{D} -lax map is a lax transformation $h : X \rightarrow Y$.

Note. When clear from context, we write $\phi_* = X(\phi)$ for a map $\phi \in \mathcal{D}$.

We spell out the details of this definition. A lax transformation $h : X \rightarrow Y$ consists of

- for each object $m \in \mathcal{D}$, a 2-functor $h_m : X(m) \rightarrow Y(m)$ and
- for each $\phi : m \rightarrow n$ in \mathcal{D} , a 2-natural transformation h_ϕ as below.

$$\begin{array}{ccc} X(m) & \xrightarrow{h_m} & Y(m) \\ \phi_* \downarrow & \lrcorner h_\phi & \downarrow \phi_* \\ X(n) & \xrightarrow{h_n} & Y(n) \end{array}$$

subject to the conditions that $h_{\text{id}_m} = 1_{h_m}$ and that $h_{\psi\phi}$ is equal to the pasting below.

$$\begin{array}{ccc} X(m) & \xrightarrow{h_m} & Y(m) \\ \phi_* \downarrow & \lrcorner h_\phi & \downarrow \phi_* \\ X(n) & \xrightarrow{h_n} & Y(n) \\ \psi_* \downarrow & \lrcorner h_\psi & \downarrow \psi_* \\ X(p) & \xrightarrow{h_p} & Y(p) \end{array}$$

One can compose lax transformations in the obvious way: given $h : X \rightarrow Y, j : Y \rightarrow Z$, we define jh by $(jh)_m = j_m \circ h_m$ and $(jh)_\phi$ as the pasting below.

$$\begin{array}{ccccc} X(m) & \xrightarrow{h_m} & Y(m) & \xrightarrow{j_m} & Z(m) \\ \phi_* \downarrow & \lrcorner h_\phi & \downarrow \phi_* & \lrcorner j_\phi & \downarrow \phi_* \\ X(n) & \xrightarrow{h_n} & Y(n) & \xrightarrow{j_n} & Z(n) \end{array}$$

It is easy to see that this composition is associative and unital.

Definition 4.3. The category $\mathcal{D}\text{-}2\text{Cat}_l$ is defined to be the category of \mathcal{D} -2-categories and \mathcal{D} -lax maps between them.

Remark 4.4. $\mathcal{D}\text{-}2\text{Cat}_l$ is actually the underlying category of the 2-category $\mathbf{Lax}(\mathcal{D}, 2\text{Cat}_2)$, the 2-category of 2-functors, lax transformations, and modifications (see Definition 4.5) from \mathcal{D} to 2Cat_2 . As every 2-natural transformation is lax, there is a canonical inclusion

$$\mathcal{D}\text{-}2\text{Cat} \hookrightarrow \mathcal{D}\text{-}2\text{Cat}_l.$$

4.2. Transformations of lax maps.

Definition 4.5. Let X, Y be \mathcal{D} -2-categories, and let $h, k : X \rightarrow Y$ be \mathcal{D} -lax maps between them. A \mathcal{D} -transformation, λ ,

$$\begin{array}{ccc} & h & \\ X & \Downarrow \lambda & Y \\ & k & \end{array}$$

is a modification between the lax transformations h and k . More precisely, λ consists of a 2-natural transformation $\lambda_m : h_m \Rightarrow k_m$ for each object $m \in \mathcal{D}$, subject to the condition that for each $\phi : m \rightarrow n$ in \mathcal{D} and $x \in X(m)$ the square below commutes.

$$\begin{array}{ccc} \phi_*(h_m(x)) & \xrightarrow{\phi_*(\lambda_m)} & \phi_*(k_m(x)) \\ h_\phi \downarrow & & \downarrow k_\phi \\ h_n(\phi_*(x)) & \xrightarrow{\lambda_n} & k_n(\phi_*(x)) \end{array}$$

As indicated by Remark 4.4, \mathcal{D} -transformations are actually the 2-cells of a 2-category.

Notation 4.6. We will let $(\mathcal{D}\text{-}2\mathcal{C}at)_2$ denote the 2-category of \mathcal{D} -2-categories, maps between them, and \mathcal{D} -transformations between those. We will let $(\mathcal{D}\text{-}2\mathcal{C}at_l)_2$ denote the 2-category of \mathcal{D} -2-categories, \mathcal{D} -lax maps between them, and \mathcal{D} -transformations between those.

Extending Remark 4.4, we then have an obvious inclusion of 2-categories

$$\text{inc} : (\mathcal{D}\text{-}2\mathcal{C}at)_2 \hookrightarrow (\mathcal{D}\text{-}2\mathcal{C}at_l)_2.$$

We give a characterization of \mathcal{D} -transformations in Lemma 4.11 using a notion of path objects adjoint to the functor $(-\times \Delta[1])$.

In the case $\mathcal{D} = \mathcal{F}$, we use the terms Γ -lax map and Γ -transformation, respectively, for \mathcal{F} -lax maps of reduced diagrams and \mathcal{F} -transformations of such. We denote by $\Gamma\text{-}2\mathcal{C}at_l$ the full subcategory of $\mathcal{F}\text{-}2\mathcal{C}at_l$ whose objects are reduced diagrams.

Definition 4.7. Let \mathcal{A} be a 2-category. Then $\mathcal{A}^{\Delta[1]}$ is defined to be the 2-category where

- objects are the arrows of \mathcal{A} ,
- a 1-cell $f \rightarrow g$ is a pair (r, s) of arrows in \mathcal{A} such that $gr = sf$, and
- a 2-cell $(r, s) \Rightarrow (r', s')$ is a pair (α, β) of 2-cells in \mathcal{A} with $\alpha : r \Rightarrow r', \beta : s \Rightarrow s'$ such that $g * \alpha = \beta * f$.

Composition and units are given componentwise in \mathcal{A} .

Proposition 4.8. For any 2-category \mathcal{A} , there is a weak equivalence $i : \mathcal{A} \rightarrow \mathcal{A}^{\Delta[1]}$, natural in 2-functors, and a pair of 2-functors $e_j : \mathcal{A}^{\Delta[1]} \rightarrow \mathcal{A}, j = 0, 1$, making the composite

$$\mathcal{A} \xrightarrow{i} \mathcal{A}^{\Delta[1]} \xrightarrow{e_0 \times e_1} \mathcal{A} \times \mathcal{A}$$

equal to the diagonal 2-functor $\mathcal{A} \rightarrow \mathcal{A} \times \mathcal{A}$. For 2-functors $F, G : \mathcal{A} \rightarrow \mathcal{B}$ and a 2-natural transformation $\alpha : F \Rightarrow G$, there is a 2-functor $\tilde{\alpha} : \mathcal{A} \rightarrow \mathcal{B}^{\Delta[1]}$ such that

$$\begin{array}{ccc} & F & \\ \mathcal{A} & \swarrow \tilde{\alpha} & \nearrow e_0 \\ & \mathcal{B}^{\Delta[1]} & \\ & \searrow G & \nearrow e_1 \\ & \mathcal{B} & \end{array}$$

commutes. This gives a bijection, natural in \mathcal{A}, \mathcal{B} between 2-functors $\tilde{\alpha}: \mathcal{A} \rightarrow \mathcal{B}^{\Delta[1]}$ and triples (F, G, α) where $F, G: \mathcal{A} \rightarrow \mathcal{B}$ and $\alpha: F \Rightarrow G$ is a 2-natural transformation.

Proof. The 2-functor i sends an object x to id_x , a 1-cell f to (f, f) , and a 2-cell α to (α, α) . The 2-functors e_0 and e_1 send an object in $\mathcal{A}^{\Delta[1]}$ to the source and target of the arrow respectively, and project correspondingly for 1- and 2-cells. Both e_0i and e_1i are the identity, and there is a 2-natural transformation $ie_0 \rightarrow \text{id}_{\mathcal{A}^{\Delta[1]}}$, so the 2-functors i, e_0, e_1 are all weak equivalences of 2-categories.

For the second claim, $\tilde{\alpha}$ sends x to α_x , f to (Ff, Gf) , and γ to $(F\gamma, G\gamma)$. The 2-naturality of α ensures that these are valid cells of $\mathcal{B}^{\Delta[1]}$, and the commutativity of the above diagram is then clear from the definition of the e_j . \square

Remark 4.9. The category 2Cat is closed symmetric monoidal with respect to the cartesian product and the hom-2-category $\mathcal{A}^{\mathcal{B}}$ consisting of 2-functors, 2-natural transformations, and modifications from \mathcal{B} to \mathcal{A} . The path 2-category $\mathcal{A}^{\Delta[1]}$ above is simply a more explicit description for the special case $\mathcal{B} = \Delta[1]$, the category $\bullet \rightarrow \bullet$ treated as a discrete 2-category. 2-functors $\mathcal{C} \rightarrow \mathcal{A}^{\Delta[1]}$ are in bijection with 2-functors $\mathcal{C} \times \Delta[1] \rightarrow \mathcal{A}$, and similarly for 2-natural transformations and modifications.

Definition 4.10. Let X be a \mathcal{D} -2-category. We define the *path \mathcal{D} -2-category* $X^{\Delta[1]}$ by

$$X^{\Delta[1]}(i) = (X(i))^{\Delta[1]}.$$

We have $e_0, e_1: X^{\Delta[1]} \rightarrow X$ given by applying the corresponding 2-functors of Proposition 4.8 levelwise.

Note. If $\mathcal{D} = \mathcal{F}$ and X is a reduced diagram, then $X^{\Delta[1]}$ is also a reduced diagram.

Lemma 4.11. Let X, Y be \mathcal{D} -2-categories. There are bijections of sets, natural in both variables, between

- i. the set whose elements are triples (h, k, λ) , where $h, k: X \rightarrow Y$ are \mathcal{D} -lax maps and $\lambda: h \rightarrow k$ is an \mathcal{D} -transformation between them; and
- ii. the set of \mathcal{D} -lax maps $X \rightarrow Y^{\Delta[1]}$.

Proof. To establish the bijection, note that given a \mathcal{D} -lax map $\tilde{\lambda}: X \rightarrow Y^{\Delta[1]}$, we get \mathcal{D} -lax maps $h = e_0\tilde{\lambda}$ and $k = e_1\tilde{\lambda}$. There is also a 2-natural transformation $\lambda_m: h_m \Rightarrow k_m$ for each object $m \in \mathcal{D}$ given by $\tilde{\lambda}_m$. The single axiom required for these λ_m to give a \mathcal{D} -transformation is a consequence of the 2-naturality of the $\tilde{\lambda}_\phi$ and the definition of 1-cells in $Y^{\Delta[1]}(n)$ being commutative squares. It is easy to check that this function from \mathcal{D} -lax maps $\tilde{\lambda}: X \rightarrow Y^{\Delta[1]}$ to triples (h, k, λ) is a bijection. \square

Note. If $\mathcal{D} = \mathcal{F}$, the statement of Lemma 4.11 also holds for Γ -transformations and Γ -lax maps into the path object.

Definition 4.12. The relative category $(\mathcal{D}\text{-}2\text{Cat}_l, \mathcal{W})$ is the category of \mathcal{D} -2-categories with \mathcal{D} -lax maps and weak equivalences the levelwise weak equivalences of 2-categories. The category $\text{ho}\mathcal{D}\text{-}2\text{Cat}_l$ is the category obtained by formally inverting these weak equivalences.

Definition 4.13. We let $(\Gamma\text{-}2\text{Cat}_l, \mathcal{W})$ denote the full subcategory of reduced diagrams in $(\mathcal{F}\text{-}2\text{Cat}_l, \mathcal{W})$ with the same weak equivalences. The category $\text{ho}\Gamma\text{-}2\text{Cat}_l$ is the category obtained by formally inverting these weak equivalences.

Remark 4.14. It is important to note that for the moment we do not know that $\text{ho}\mathcal{D}\text{-}2\text{Cat}_l$ is locally small as the localization process could have produced a proper class of maps between some pair of objects. The isomorphism of homotopy categories in Theorem 4.37 shows that $\text{ho}\mathcal{D}\text{-}2\text{Cat}_l$ is indeed locally small.

We point out that the laxity in a \mathcal{D} -lax map occurs at the level of functoriality with respect to the maps in \mathcal{D} , not at the level of the maps between the individual 2-categories: those are still strict 2-functors. Thus the \mathcal{D} -lax structure does not play a role in deciding whether or not a map is a levelwise weak equivalence.

Corollary 4.15. *If $\lambda: h \rightarrow k$ is a \mathcal{D} -transformation, then $[h] = [k]$ in $\text{ho}\mathcal{D}\text{-2Cat}_l$.*

Proof. This is a consequence of Proposition 4.8 and a standard path object argument ([Hir03, 5.11]). Since $i: X \rightarrow X^{\Delta^{[1]}}$ is a levelwise weak equivalence and $e_j i = \text{id}$ for $j = 0, 1$, then $[i]$ is an isomorphism in $\text{ho}\mathcal{D}\text{-2Cat}_l$ so $[e_0 i] = [e_1 i]$ implies $[e_0] = [e_1]$. Therefore

$$[h] = [e_0 \tilde{\alpha}] = [e_0][\tilde{\alpha}] = [e_1][\tilde{\alpha}] = [e_1 \tilde{\alpha}] = [k].$$

□

Note. If $\mathcal{D} = \mathcal{F}$, then Corollary 4.15 shows that a Γ -transformation induces an equality in $\text{ho}\Gamma\text{-2Cat}_l$.

We now return to the subject of primary interest, namely Γ -2-categories and their homotopy theory.

Definition 4.16. A Γ -lax map $h: X \rightarrow Y$ is a *stable equivalence* if the function

$$h^*: \text{ho}\Gamma\text{-2Cat}_l(Y, Z) \rightarrow \text{ho}\Gamma\text{-2Cat}_l(X, Z)$$

is an isomorphism for every very special Γ -2-category Z .

Definition 4.17. The relative category $(\Gamma\text{-2Cat}_l, S)$ is the category of Γ -2-categories with Γ -lax maps and weak equivalences the stable equivalences of Γ -2-categories. The category $\text{Ho}\Gamma\text{-2Cat}_l$ is the category obtained by formally inverting these weak equivalences.

4.3. Grothendieck constructions. In this section we describe the Grothendieck construction for diagrams of 2-categories

$$X: \mathcal{D} \rightarrow \text{2Cat}.$$

This can be regarded as an enrichment of the standard construction for diagrams of categories, or as a specialization of the construction given in [CCG11] for lax diagrams of bicategories. We use the notation of [CCG11].

Definition 4.18. The (covariant) 2-categorical Grothendieck construction on X is a 2-category

$$\mathcal{D}/X$$

which has objects, 1-cells, and 2-cells given by pairs as below, where we use square brackets for tuples in the Grothendieck construction.

$$\begin{array}{ccc} & [\phi, f] & \\ [m, x] & \begin{array}{c} \swarrow \\ \Downarrow [\phi, \alpha] \\ \searrow \end{array} & [n, y] \\ & [\phi, g] & \end{array}$$

Here $\phi: m \rightarrow n$ is a morphism in \mathcal{D} , $x \in X(m)$, $y \in X(n)$, and f, g, α are 1- and 2-cells in $X(n)$ as below.

$$\begin{array}{ccc} & f & \\ \phi_* x & \begin{array}{c} \swarrow \\ \Downarrow \alpha \\ \searrow \end{array} & y \\ & g & \end{array}$$

Composition of 1-cells in \mathcal{D}/X

$$[m, x] \xrightarrow{[\phi, f]} [n, y] \xrightarrow{[\psi, g]} [p, z]$$

is given by $[\psi\phi, g \circ \psi_* f]$, which corresponds to the following composite in $X(p)$

$$\psi_* \phi_* x \xrightarrow{\psi_* f} \psi_* y \xrightarrow{g} z.$$

Horizontal composition of 2-cells is given similarly. Vertical composition of 2-cells is given by composing vertically the second component in $X(n)$. Both compositions are described explicitly in [CCG11].

Remark 4.19. The laxness direction for what we have called a \mathcal{D} -lax map X to Y is what would be called in [CCG11] a lax map from X^{op} to Y^{op} – the diagram categories where the direction of 1-cells is reversed in each $X(m)$ and $Y(m)$. Our definitions of the Grothendieck construction and rectification therefore also differ from those of [CCG11] in the direction of 1-cells. These are identified by interchanging X with X^{op} .

To see that the Grothendieck construction defines a functor, one can specialize the work in [CCG11] or take a $\mathcal{C}\text{at}$ -enrichment of [Str72] to obtain the following.

Proposition 4.20. *The Grothendieck construction defines a functor*

$$(\mathcal{D}\mathcal{J}-): \mathcal{D}\text{-}2\mathcal{C}\text{at}_l \rightarrow 2\mathcal{C}\text{at}.$$

Remark 4.21. It will be useful for us to have a description of $\mathcal{D}\mathcal{J}h$ for a \mathcal{D} -lax map $h: X \rightarrow Y$. The functor $\mathcal{D}\mathcal{J}h: \mathcal{D}\mathcal{J}X \rightarrow \mathcal{D}\mathcal{J}Y$ is given explicitly as follows:

On 0-cells

$$(\mathcal{D}\mathcal{J}h)[m, x] = [m, h_m(x)].$$

On 1-cells

$$(\mathcal{D}\mathcal{J}h)[\phi, f] = [\phi, h_n(f) \circ h_\phi].$$

On 2-cells

$$(\mathcal{D}\mathcal{J}h)[\phi, \alpha] = [\phi, h_n(\alpha) * 1_{h_\phi}].$$

These are given by the following 0-, 1-, and 2-cells in $Y(n)$.

$$\begin{array}{ccc} \phi_* h_m(x) & \xrightarrow{h_\phi} & h_n(\phi_* x) \\ & & \swarrow \Downarrow h_n(\alpha) \quad \searrow h_n(f) \\ & & h_n(y) \\ & & \downarrow h_n(g) \end{array}$$

4.4. Symmetric monoidal diagrams. In this section we give a basic theory of symmetric monoidal diagrams and monoidal lax maps in $2\mathcal{C}\text{at}$. The main result here is that Grothendieck constructions of symmetric monoidal diagrams and monoidal lax maps are, respectively, permutative 2-categories and strict maps of such. In Section 5.3 we apply this theory to produce permutative 2-categories from Γ -2-categories.

Definition 4.22. Let $(\mathcal{D}, \oplus, e, \beta)$ be a permutative category. A *symmetric monoidal \mathcal{D} -2-category* (X, ν) is a symmetric monoidal functor

$$(X, \nu): (\mathcal{D}, \oplus) \rightarrow (2\mathcal{C}\text{at}, \times).$$

In particular, this means that there is a collection of 2-functors

$$\nu_{m,p}: X(m) \times X(p) \rightarrow X(m \oplus p)$$

that are natural with respect to maps in $\mathcal{D} \times \mathcal{D}$. There is also a 2-functor $\nu_e: * \rightarrow X(e)$, where $*$ denotes the terminal 2-category.

Definition 4.23. Let (X, ν) and (Y, μ) be symmetric monoidal \mathcal{D} -2-categories. A \mathcal{D} -lax map $h: X \rightarrow Y$ is a *monoidal \mathcal{D} -lax map* if the following conditions hold.

i. The diagram below commutes.

$$(4.24) \quad \begin{array}{ccc} & & X(e) \\ & \nearrow v_e & \downarrow h_e \\ * & & \\ & \searrow \mu_e & \downarrow \\ & & Y(e) \end{array}$$

ii. For all $m, p \in \mathcal{D}$ the square below commutes.

$$(4.25) \quad \begin{array}{ccc} X(m) \times X(p) & \xrightarrow{v_{m,p}} & X(m \oplus p) \\ h_m \times h_p \downarrow & & \downarrow h_{m \oplus p} \\ Y(m) \times Y(p) & \xrightarrow{\mu_{m,p}} & Y(m \oplus p) \end{array}$$

iii. For all $\phi: m \rightarrow n$ and $\psi: p \rightarrow q$ in \mathcal{D} we have the following equality of pasting diagrams.

$$\begin{array}{ccccc} X(m) \times X(p) & \xrightarrow{v_{m,p}} & X(m \oplus p) & & \\ h_m \times h_p \swarrow & = & h_{m \oplus p} \swarrow & & (\phi \oplus \psi)_* \searrow \\ Y(m) \times Y(p) & \xrightarrow{\mu_{m,p}} & Y(m \oplus p) & \xrightarrow{h_{\phi \oplus \psi}} & X(n \oplus q) \\ \phi_* \times \psi_* \searrow & = & (\phi \oplus \psi)_* \swarrow & & h_{n \oplus q} \swarrow \\ Y(n) \times Y(q) & \xrightarrow{\mu_{n,q}} & Y(n \oplus q) & & \end{array}$$

$$(4.26) \quad \begin{array}{ccccc} & & & & \parallel \\ & & & & \\ X(m) \times X(p) & \xrightarrow{v_{m,p}} & X(m \oplus p) & & \\ h_m \times h_p \swarrow & & \swarrow \phi_* \times \psi_* & = & (\phi \oplus \psi)_* \searrow \\ Y(m) \times Y(p) & \xrightarrow{h_{\phi} \times h_{\psi}} & X(n) \times X(q) & \xrightarrow{v_{n,q}} & X(n \oplus q) \\ \phi_* \times \psi_* \searrow & & h_n \times h_q \swarrow & = & h_{n \oplus q} \swarrow \\ Y(n) \times Y(q) & \xrightarrow{\mu_{n,q}} & Y(n \oplus q) & & \end{array}$$

Remark 4.27. Because the cartesian product of 2-categories is strictly functorial, the composite of monoidal \mathcal{D} -lax maps is again monoidal. The collection of symmetric monoidal \mathcal{D} -2-categories and monoidal \mathcal{D} -lax maps therefore forms a category with a faithful forgetful functor to $\mathcal{D}\text{-}2\text{Cat}_l$.

Definition 4.28. Given a permutative category $\mathcal{D} = (\mathcal{D}, \oplus, e, \beta)$, let $(\mathcal{D}, \oplus)\text{-}2\text{Cat}_l$ denote the category of symmetric monoidal \mathcal{D} -2-categories and monoidal \mathcal{D} -lax maps.

Proposition 4.29. Let $(\mathcal{D}, \oplus, e, \beta)$ be a permutative category and let (X, v) be a symmetric monoidal \mathcal{D} -2-category. Then the Grothendieck construction $\mathcal{D}\text{/}X$ is a permutative 2-category.

Proof. The monoidal product on objects of $\mathcal{D}fX$ is defined as

$$([m, x], [p, y]) \mapsto [m \oplus p, \nu_{m, p}(x, y)].$$

The product of a pair of 1-morphisms $([\phi, f], [\psi, g])$ is given by the 1-morphism whose first coordinate is $\phi \oplus \psi$ and whose second coordinate is the 1-morphism

$$(\phi \oplus \psi)_* \nu_{m, p}(x, y) = \nu_{n, q}(\phi_* x, \psi_* y) \xrightarrow{\nu_{n, q}(f, g)} \nu_{n, q}(x', y').$$

Pairs of 2-morphisms are treated similarly.

It is routine to check that this product is a 2-functor. Since \mathcal{D} is a permutative category and (X, ν) is a symmetric monoidal functor, it is straightforward to see that this 2-functor defines a permutative structure on $\mathcal{D}fX$ with monoidal unit given by $[e, \nu_e(*)]$ and symmetry transformation given by

$$(4.30) \quad [(m \oplus n, \nu_{m, n}(x, y))] \xrightarrow{[\beta_{m, n}, \text{id}]} [n \oplus m, \nu_{n, m}(y, x)]. \quad \square$$

Proposition 4.31. *Let \mathcal{D} be a permutative category and let $h: X \rightarrow Y$ be a monoidal \mathcal{D} -lax map of symmetric monoidal \mathcal{D} -2-categories. Then*

$$\mathcal{D}f h: \mathcal{D}fX \rightarrow \mathcal{D}fY$$

is a strict symmetric monoidal 2-functor between permutative 2-categories. This construction gives the assignment on morphisms of a functor

$$\mathcal{D}f -: (\mathcal{D}, \oplus)\text{-2Cat}_l \rightarrow \text{Perm2Cat}.$$

Proof. The fact that $\mathcal{D}f h$ preserves the monoidal structure strictly at the level of objects follows from the condition in Display (4.25). Likewise, the condition in Display (4.26) shows that $\mathcal{D}f h$ preserves the product on 1- and 2-cells strictly. The fact that $\mathcal{D}f h$ preserves the unit strictly follows from Display (4.24). To see that $\mathcal{D}f h$ strictly preserves the symmetry, apply the description of $\mathcal{D}f h$ on 1-cells from Remark 4.21 to the formula for the symmetry given in Display (4.30). \square

4.5. From lax maps to spans. In this section we describe a construction, inspired by that of [Man10], which allows us to replace a Γ -lax map with a weakly-equivalent span of strict Γ -maps. This is a key ingredient in our proof of Theorem 1.1, but we also use it in this section to prove that the homotopy categories of $\mathcal{D}\text{-2Cat}$ and $\mathcal{D}\text{-2Cat}_l$ are isomorphic. Specializing to reduced diagrams on \mathcal{F} , we prove the following isomorphisms of categories (see Corollaries 4.46 and 4.48):

$$\begin{aligned} \text{ho}\Gamma\text{-2Cat}_l &\cong \text{ho}\Gamma\text{-2Cat}, \\ \text{Ho}\Gamma\text{-2Cat}_l &\cong \text{Ho}\Gamma\text{-2Cat}. \end{aligned}$$

Definition 4.32. Let $k: X \rightarrow Z$ be a \mathcal{D} -lax map. For each $m \in \mathcal{D}$, define a 2-category $Ek(m)$ as the following pullback in 2Cat .

$$\begin{array}{ccc} Ek(m) & \xrightarrow{\bar{\nu}} & Z^{\Delta[1]}(m) \\ \omega \downarrow & & \downarrow e_1 \\ X(m) & \xrightarrow{k} & Z(m) \end{array}$$

Thus $Ek(m)$ has 0-cells given by triples (x, f, a) where x is an object in $X(m)$, a is an object in $Z(m)$, and

$$a \xrightarrow{f} k(x)$$

is a 1-cell in $Z(m)$.

A 1-cell from (x, f, a) to (y, g, b) is given by a pair (s, r) where $s: x \rightarrow y$ and $r: a \rightarrow b$ are 1-cells in $X(m)$ and $Z(m)$, respectively, such that the diagram

$$\begin{array}{ccc} a & \xrightarrow{r} & b \\ f \downarrow & & \downarrow g \\ k(x) & \xrightarrow{k(s)} & k(y) \end{array}$$

commutes. Similarly, a 2-cell from (s, r) to (s', r') is a pair (β, α) of 2-cells $\beta: s \Rightarrow s'$ and $\alpha: r \Rightarrow r'$ in $X(m)$ and $Z(m)$, respectively, such that

$$\begin{array}{ccc} a & \xrightarrow{r} & b \\ \downarrow \alpha & \nearrow r' & \\ k(x) & \xrightarrow{k(s')} & k(y) \end{array} = \begin{array}{ccc} a & \xrightarrow{r} & b \\ f \downarrow & & \downarrow g \\ k(x) & \xrightarrow{k(s)} & k(y) \\ \downarrow k(\beta) & \nearrow k(s') & \\ k(x) & \xrightarrow{k(s')} & k(y) \end{array}$$

These cells are depicted as follows

$$\begin{array}{ccc} a & \xrightarrow{r} & b \\ \downarrow \alpha & \nearrow r' & \\ k(x) & \xrightarrow{k(s)} & k(y) \\ \downarrow k(\beta) & \nearrow k(s') & \\ k(x) & \xrightarrow{k(s')} & k(y) \end{array}$$

Vertical and horizontal composition are given by the corresponding compositions in $X(m)$ and $Z(m)$. The 2-functors ω and $\bar{\nu}$ are given by projection onto the first and second components, respectively.

For a map $\phi: m \rightarrow n$ in \mathcal{D} , there is a 2-functor

$$\phi_*: Ek(m) \rightarrow Ek(n)$$

that sends the data above to the data represented by the following diagram.

$$(4.33) \quad \begin{array}{ccc} \phi_* a & \xrightarrow{\phi_* r} & \phi_* b \\ \phi_*(f) \downarrow & \downarrow \phi_* \alpha & \downarrow \phi_*(g) \\ \phi_* k(x) & \xrightarrow{\phi_* r'} & \phi_* k(y) \\ k_\phi \downarrow & \downarrow k\phi_*(s) & \downarrow k_\phi \\ k\phi_*(x) & \xrightarrow{\phi_* k(\beta)} & k\phi_*(y) \\ & \downarrow k\phi_*(s') & \\ & k\phi_*(s') & \end{array}$$

One can verify that $(\psi\phi)_* = \psi_*\phi_*$ directly from the description in Display (4.33) using the equality of $k_{\psi\phi}$ with the pasting of k_ψ and k_ϕ (see Definition 4.2). This proves the following proposition.

Proposition 4.34. *The 2-categories $Ek(m)$ assemble to make Ek a \mathcal{D} -2-category.*

Proposition 4.35. *The \mathcal{D} -2-category Ek fits in a commuting diagram in \mathcal{D} -2Cat_l shown below. The map ω and the horizontal composite v are both strict \mathcal{D} -maps.*

$$\begin{array}{ccccc}
 & & v & & \\
 & \nearrow \bar{v} & \curvearrowright & \searrow e_0 & \\
 Ek & \longrightarrow & Z^{\Delta[1]} & \longrightarrow & Z \\
 \downarrow \omega & & \downarrow e_1 & & \\
 X & \xrightarrow{k} & Z & &
 \end{array}$$

Proof. For a map $\phi: m \rightarrow n$ in \mathcal{D} it is immediate that ω strictly commutes with ϕ_* . The laxity of k provides 1-cells $\phi_* \bar{v}(x, f, a) \rightarrow \bar{v} \phi_*(x, f, a)$ in $Z^{\Delta[1]}(n)$ as below:

$$\phi_* \bar{v}(x, f, a) = \phi_*(f) \xrightarrow{(\text{id}_{\phi_*(a)}, k_\phi)} k_\phi \circ \phi_*(f) = \bar{v} \phi_*(x, f, a).$$

These are 2-natural because k_ϕ is 2-natural, and the conditions for identity 1-cells and pasting on composites are immediate from those of the k_ϕ .

The map e_0 is given by projecting to the source of each cell in $Z^{\Delta[1]}$, and, as the laxity of \bar{v} is concentrated in the target components, the composite $v = e_0 \bar{v}$ is a strict \mathcal{D} -map. \square

Proposition 4.36. *The map ω is a levelwise left 2-adjoint and hence is a levelwise weak equivalence.*

Proof. Consider the 2-functor $i: X(m) \rightarrow Ek(m)$ given by

$$\begin{aligned}
 x &\mapsto (x, \text{id}_{k(x)}, k(x)) \\
 s &\mapsto (s, k(s)) \\
 \beta &\mapsto (\beta, k(\beta)).
 \end{aligned}$$

It is easy to check that this assignment is indeed a 2-functor that lands in $Ek(m)$ and that ωi is the identity on $X(m)$. Note that for an object (x, f, a) in $Ek(m)$, the pair (id_x, f) is a 1-cell from (x, f, a) to $(x, \text{id}_{k(x)}, k(x))$. We leave to the reader to check that these 1-cells give a 2-natural transformation from the identity on $Ek(m)$ to $i\omega$.

The triangle identities are immediate: First note that $\omega(\text{id}_x, f) = \text{id}_x$ and therefore applying ω to the counit is the identity. Second, the component of the counit on $i(x)$ is $(\text{id}_x, \text{id}_{k(x)})$. \square

Recall that $\text{ho}\mathcal{D}\text{-2Cat}$, $\text{ho}\mathcal{D}\text{-2Cat}_l$ denote the categories obtained by formally inverting the levelwise weak equivalences of \mathcal{D} -2-categories in the categories $\mathcal{D}\text{-2Cat}$, $\mathcal{D}\text{-2Cat}_l$, respectively. Since the laxity of a map plays no role in determining whether it is or is not a levelwise weak equivalence, the inclusion

$$\mathcal{D}\text{-2Cat} \hookrightarrow \mathcal{D}\text{-2Cat}_l$$

both preserves and reflects levelwise equivalences, and therefore induces a functor on homotopy categories

$$\text{ho}\mathcal{D}\text{-2Cat} \rightarrow \text{ho}\mathcal{D}\text{-2Cat}_l.$$

Theorem 4.37. *The functor $\text{ho}\mathcal{D}\text{-2Cat} \rightarrow \text{ho}\mathcal{D}\text{-2Cat}_l$ is an isomorphism of categories.*

Proof. Let $k: X \rightarrow Y$ be a \mathcal{D} -lax map, and consider the commutative diagram of Proposition 4.35. Then we have the following calculation in $\text{ho}\mathcal{D}\text{-2Cat}_l$ using that $[e_0] = [e_1]$ as in Corollary 4.15.

$$[k\omega] = [e_1 \bar{v}] = [e_0 \bar{v}] = [v]$$

Thus $[k] = [\nu][\omega]^{-1}$ in $\text{ho}\mathcal{D}\text{-2Cat}_l$ since, by Proposition 4.36, ω is a levelwise weak equivalence. Now levelwise weak equivalences satisfy the 2 out of 3 property, so if k is a levelwise equivalence then ν is also. Therefore $[k]^{-1} = [\omega][\nu]^{-1}$ in $\text{ho}\mathcal{D}\text{-2Cat}_l$. Since every morphism in $\text{ho}\mathcal{D}\text{-2Cat}_l$ is a composite of the form

$$[k_1]^{-1}[k_2][k_3]^{-1}\cdots[k_{2n}]$$

where the k_{2i+1} 's are all levelwise equivalences, the function on hom-sets

$$\text{ho}\mathcal{D}\text{-2Cat}(X, Y) \rightarrow \text{ho}\mathcal{D}\text{-2Cat}_l(X, Y)$$

is seen to be surjective by replacing the $[k_{2i}]$ with $[\nu_{2i}][\omega_{2i}]^{-1}$ and the $[k_{2i+1}]^{-1}$ with $[\omega_{2i+1}][\nu_{2i+1}]^{-1}$. Injectivity then follows from the fact that performing this procedure on a morphism in the image of $\text{ho}\mathcal{D}\text{-2Cat}(X, Y) \rightarrow \text{ho}\mathcal{D}\text{-2Cat}_l(X, Y)$ is clearly the identity. Since these categories have the same objects, the map is therefore an isomorphism. \square

Proposition 4.38. *Given a composable pair of \mathcal{D} -lax maps*

$$X \xrightarrow{h} Y \xrightarrow{j} Z$$

there are \mathcal{D} -lax maps

$$Eh \xrightarrow{j_E} E(jh) \xrightarrow{h_E} E(j)$$

induced by maps on levelwise pullbacks. The map j_E , resp. h_E , is a strict \mathcal{D} -map if j , resp. h , is a strict \mathcal{D} -map.

Proposition 4.39. *Given a parallel pair of \mathcal{D} -lax maps and a \mathcal{D} -transformation*

$$\begin{array}{ccc} & h & \\ X & \swarrow \downarrow \lambda \searrow & Y \\ & k & \end{array}$$

there is a strict \mathcal{D} -map

$$E\lambda: Eh \rightarrow Ek.$$

This defines a functor

$$E: \mathbf{Lax}(\mathcal{D}, \text{2Cat}_2)(X, Y) \rightarrow \mathcal{D}\text{-2Cat}$$

from the category of \mathcal{D} -lax maps and \mathcal{D} -transformations between X and Y to the category of \mathcal{D} -2-categories and strict \mathcal{D} -maps.

Proof. The map $E\lambda$ is given on 0-cells of $Eh(m)$ by $E\lambda(x, f, a) = (x, \lambda_m(x) \circ f, a)$ and is the identity on 1- and 2-cells. The 2-naturality of λ_m ensures that this is a well-defined map to $Eh(m)$, and the condition $\lambda_n(x) \circ h_\phi(x) = k_\phi(x) \circ \phi_*(\lambda_m(x))$ of Definition 4.5 ensures that this is a strict \mathcal{D} -map.

It is immediate that this construction is functorial with respect to \mathcal{D} -lax maps and produces the identity map if λ is the identity. \square

For a square of \mathcal{D} -lax maps with a \mathcal{D} -transformation between the two composites

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ h \downarrow & \swarrow \lambda \searrow & \downarrow k \\ Y & \xrightarrow{j} & W \end{array}$$

we combine Propositions 4.38 and 4.39 to obtain the following maps:

$$(4.40) \quad \begin{array}{ccccc} Eh & \xrightarrow{j_E} & E(jh) & \xrightarrow{h_E} & Ej \\ & & \downarrow E\lambda & & \\ Ei & \xrightarrow{k_E} & E(ki) & \xrightarrow{i_E} & Ek \end{array}$$

When the maps h and k are strict and λ is the identity, we can say more. To this end, let

$$\mathcal{D}\text{-}2\text{Cat}_{\text{lax},\text{str}}^{\bullet\rightarrow\bullet}$$

denote the arrow category whose objects are \mathcal{D} -lax maps $X \xrightarrow{i} Z$ and whose morphisms are strict maps making the corresponding squares commute:

$$(4.41) \quad \begin{array}{ccc} X & \xrightarrow{i} & Z \\ h \downarrow & & \downarrow k \\ Y & \xrightarrow{j} & W \end{array}$$

Composition is given by stacking and pasting vertically, along the strict \mathcal{D} -maps.

Definition 4.42. Let \mathcal{E} be a category.

- i. A *span* in \mathcal{E} from X to Y is a diagram of the form $X \leftarrow A \rightarrow Y$.
- ii. A *map of spans* from $X \leftarrow A \rightarrow Y$ to $X' \leftarrow A' \rightarrow Y'$ consists of maps $f: X \rightarrow X', g: A \rightarrow A', h: Y \rightarrow Y'$ making the two squares commute.

$$\begin{array}{ccccc} X & \longleftarrow & A & \longrightarrow & Y \\ f \downarrow & & \downarrow g & & \downarrow h \\ X' & \longleftarrow & A' & \longrightarrow & Y' \end{array}$$

- iii. The category $\text{Span}(\mathcal{E})$ has objects spans in \mathcal{E} , and maps of spans between them.

Remark 4.43. It is clear that $\text{Span}(\mathcal{E})$ is merely the functor category $[\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{E}]$.

We thus have the following refinement of Display (4.40).

Corollary 4.44. *The construction E defines a functor*

$$\mathcal{D}\text{-}2\text{Cat}_{\text{lax},\text{str}}^{\bullet\rightarrow\bullet} \longrightarrow \text{Span}(\mathcal{D}\text{-}2\text{Cat})$$

given on objects by sending a lax map $i: X \rightarrow Z$ to

$$X \xleftarrow{\omega} Ei \xrightarrow{\nu} Z.$$

Proof. Given a commuting square as in Display (4.41), we have the composite \mathcal{D} -map

$$Ei \xrightarrow{k_E} E(ki) = E(jh) \xrightarrow{h_E} Ej.$$

This is a strict \mathcal{D} -map because h and k are strict. One verifies immediately that this map commutes with the structure maps ω and ν .

A lengthy but straightforward check verifies that this construction is functorial with respect to the morphisms of $\text{Span}(\mathcal{D}\text{-}2\text{Cat})$. \square

Remark 4.45. The category $\mathcal{D}\text{-}2\text{Cat}_{\text{lax},\text{str}}^{\bullet\rightarrow\bullet}$ arises naturally in the following way. There is a double category with objects \mathcal{D} -2-categories, vertical arrows strict \mathcal{D} -maps, horizontal arrows lax- \mathcal{D} -maps, and cells given by commutative squares. We can view this double category as a category internal to categories by taking the category of objects to be $\mathcal{D}\text{-}2\text{Cat}_1$ and then $\mathcal{D}\text{-}2\text{Cat}_{\text{lax},\text{str}}^{\bullet\rightarrow\bullet}$ appears as the category of arrows.

We now concentrate on the case of Γ -2-categories. Recall that Γ - $2\mathcal{C}at$ and Γ - $2\mathcal{C}at_l$ denote the full subcategories of \mathcal{F} - $2\mathcal{C}at$ and \mathcal{F} - $2\mathcal{C}at_l$, respectively, given by reduced diagrams. Likewise, we let Γ - $2\mathcal{C}at_{\text{lax},\text{str}}^{\bullet \rightarrow \bullet}$ denote the full subcategory of \mathcal{F} - $2\mathcal{C}at_{\text{lax},\text{str}}^{\bullet \rightarrow \bullet}$ whose objects are lax maps between Γ -2-categories.

Note that if $k: X \rightarrow Z$ is a Γ -lax map of Γ -2-categories, then the \mathcal{F} -2-categories $Z^{\Delta[1]}$ and Ek are reduced. This leads to the following refinements of our results above.

Corollary 4.46. *The functor $\text{ho}\Gamma$ - $2\mathcal{C}at \rightarrow \text{ho}\Gamma$ - $2\mathcal{C}at_l$ is an isomorphism of categories.*

Proof. When considering only Γ -2-categories, the diagram of Proposition 4.35 is a diagram of Γ -2-categories. The result then follows by repeating the rest of the proof of Theorem 4.37. \square

Corollary 4.47. *The construction E defines a functor*

$$\Gamma\text{-}2\mathcal{C}at_{\text{lax},\text{str}}^{\bullet \rightarrow \bullet} \longrightarrow \text{Span}(\Gamma\text{-}2\mathcal{C}at).$$

Proof. This is immediate from Corollary 4.44. \square

Corollary 4.48. *A strict map $h: X \rightarrow Y$ of Γ -2-categories is a stable equivalence in the sense of Definition 2.18 if and only if, when considered as a lax map, it is a stable equivalence in the sense of Definition 4.16. In particular, the inclusion Γ - $2\mathcal{C}at \hookrightarrow \Gamma$ - $2\mathcal{C}at_l$ preserves and reflects stable equivalences, so induces a functor of stable homotopy categories*

$$\text{Ho}\Gamma\text{-}2\mathcal{C}at \rightarrow \text{Ho}\Gamma\text{-}2\mathcal{C}at_l.$$

This functor is an isomorphism.

Proof. The first statement is obvious by Corollary 4.46, and the second is immediate from the first. One then uses the same argument as in Theorem 4.37 to show that the functors on stable homotopy categories are isomorphisms. \square

Corollary 4.49. *Let $h: X \rightarrow Z$ be a Γ -lax map of Γ -2-categories and consider the construction*

$$X \xleftarrow{\omega} Eh \xrightarrow{\nu} Z$$

of Proposition 4.35. Then h is a levelwise, resp. stable, equivalence if and only if ν is a levelwise, resp. stable, equivalence.

Proof. We have

$$h\omega = e_1 \bar{\nu}$$

and

$$\nu = e_0 \bar{\nu}$$

by construction. We know that ω , e_0 , and e_1 are all levelwise equivalences and that both ω and ν are strict maps.

The claim for levelwise equivalences is immediate by 2 out of 3 for levelwise equivalences. Using 2 out of 3 for stable equivalences shows that h is a stable equivalence if and only if ν is a stable equivalence in Γ - $2\mathcal{C}at_l$. The result then follows by Corollary 4.48. \square

5. PERMUTATIVE 2-CATEGORIES FROM Γ -2-CATEGORIES

In this section we use the Grothendieck construction of Section 4.3 to build permutative 2-categories from Γ -2-categories. We first show in Section 5.1 that the category of Γ -2-categories admits a full and faithful embedding to a category of symmetric monoidal diagrams on a permutative category \mathcal{A} . We show that this extends to a functor on Γ -lax transformations in Section 5.2 and use this to give a functor

$$P: \Gamma\text{-}2\mathcal{C}at_l \rightarrow \text{Perm2Cat}$$

in Section 5.3.

Notation 5.1. For symmetric monoidal categories \mathcal{A} and \mathcal{B} , let $\mathbf{SOp\! lax}(\mathcal{A}, \mathcal{B})$ denote the category of symmetric oplax monoidal functors from \mathcal{A} to \mathcal{B} and monoidal transformations between them. Let $\mathbf{SOp\! lax}_{\text{nor}}(\mathcal{A}, \mathcal{B})$ denote the full subcategory of *normal* symmetric oplax monoidal functors, i.e., those which strictly preserve the unit.

5.1. Γ and the category \mathcal{A} . There is a monad on the category of sets which adds a basepoint. Restricting this monad to a skeleton of finite sets, \mathcal{N} , the category \mathcal{F} is isomorphic to both its category of algebras and its Kleisli category. Using the general theory of Kleisli categories, one can check the following universal property of Γ (see Leinster [Lei00] for a full proof).

Theorem 5.2. *Let \mathcal{C} be any category with finite products and terminal object 1. Then there is an isomorphism of categories*

$$[\mathcal{F}, \mathcal{C}] \cong \mathbf{SOp\! lax}((\mathcal{N}, +, \emptyset), (\mathcal{C}, \times, *)).$$

This restricts to an isomorphism between the full subcategories of reduced diagrams and normal functors:

$$\Gamma\text{-}\mathcal{C} = [\mathcal{F}, \mathcal{C}]_{\text{red}} \cong \mathbf{SOp\! lax}_{\text{nor}}((\mathcal{N}, +, \emptyset), (\mathcal{C}, \times, *)).$$

We can even go a bit further using the standard techniques of 2-dimensional algebra [Lac02a]. Recall that for a 2-monad T on a 2-category K , we have a variety of 2-categories of algebras (by which we always mean strict algebras).

- There is the 2-category K^T of T -algebras, strict T -algebra morphisms, and algebra 2-cells. This 2-category is the enriched category of algebras [Lac02a] and is often denoted $T\text{-}\mathbf{Alg}_s$.
- There is the 2-category $T\text{-}\mathbf{Alg}$ of T -algebras, pseudo- T -algebra morphisms, and algebra 2-cells.
- There is the 2-category $T\text{-}\mathbf{Alg}_l$ of T -algebras, lax T -algebra morphisms, and algebra 2-cells. Reversing the direction of the structure cells for the morphisms will produce oplax T -algebra morphisms and the 2-category $T\text{-}\mathbf{Alg}_{op}$.

These 2-categories come with canonical inclusions as displayed below.

$$\begin{array}{ccc} & & T\text{-}\mathbf{Alg}_l \\ & \nearrow & \searrow \\ T\text{-}\mathbf{Alg}_s & \longrightarrow & T\text{-}\mathbf{Alg} \\ & \searrow & \nearrow \\ & & T\text{-}\mathbf{Alg}_{op} \end{array}$$

Under certain conditions on K and T (see [Lac02a]), the inclusions $T\text{-}\mathbf{Alg}_s \hookrightarrow T\text{-}\mathbf{Alg}_m$ will have left 2-adjoints; here $T\text{-}\mathbf{Alg}_m$ indicates any of the 2-categories $T\text{-}\mathbf{Alg}$, $T\text{-}\mathbf{Alg}_l$, $T\text{-}\mathbf{Alg}_{op}$, although the particular hypotheses vary depending on which variant is used. We will generically denote this left adjoint by Q .

When $K = \mathbf{Cat}$ and T is the 2-monad for symmetric monoidal categories, we write $\mathbf{SymMonCat}$, with no subscript, for $T\text{-}\mathbf{Alg}$ (the pseudo-morphism variant). We have a left 2-adjoint $Q : \mathbf{SymMonCat}_{op} \rightarrow \mathbf{SymMonCat}_s$, and in particular an isomorphism of hom-categories

$$\mathbf{SymMonCat}_{op}(\mathcal{A}, \mathcal{B}) \cong \mathbf{SymMonCat}_s(Q\mathcal{A}, \mathcal{B}).$$

Now $\mathbf{SymMonCat}_{op}(\mathcal{A}, \mathcal{B})$ is exactly the same category that we denoted as $\mathbf{SOp\! lax}(\mathcal{A}, \mathcal{B})$ above, so combining these results with Theorem 5.2 we obtain isomorphisms (with the

monoidal structure on \mathcal{C} being the cartesian one, when required)

$$(5.3) \quad [\mathcal{F}, \mathcal{C}] \cong \mathbf{SOpLax}(\mathcal{N}, \mathcal{C}) \cong \text{SymMonCat}_s(Q(\mathcal{N}), \mathcal{C}).$$

When $K = \text{Cat}_*$, the 2-category of pointed categories, there is another 2-monad T whose 2-category of strict algebras and strict algebra maps is again SymMonCat_s ; this 2-monad can be seen as a quotient of the one on Cat which identifies the basepoint as the unit for the monoidal structure. By construction, any algebra morphism for T will necessarily preserve the basepoint, and hence the unit, so is a normal functor. In particular, $T\text{-Alg}_{op}$ is then the 2-category of symmetric monoidal categories, normal oplax symmetric monoidal functors, and monoidal transformations. This T also preserves filtered colimits, and hence we have a left-adjoint $Q' : \text{SymMonCat}_{nop} \rightarrow \text{SymMonCat}_s$. Combining this with Theorem 5.2 we have

$$(5.4) \quad \Gamma\text{-}\mathcal{C} = [\mathcal{F}, \mathcal{C}]_{\text{red}} \cong \mathbf{SOpLax}_{\text{nor}}(\mathcal{N}, \mathcal{C}) \cong \text{SymMonCat}_s(Q'(\mathcal{N}), \mathcal{C})$$

Finally, we note that every symmetric monoidal category is equivalent as such to a permutative category [Isb69], i.e., a symmetric monoidal category in which the underlying monoidal structure is strict. This equivalence is an internal equivalence in the 2-category of symmetric monoidal categories, symmetric monoidal pseudofunctors, and monoidal transformations, in other words in the 2-category SymMonCat . Thus we can find a permutative category $\mathcal{A} \simeq Q'(\mathcal{N})$, and therefore obtain an equivalence of categories

$$\text{SymMonCat}(Q'(\mathcal{N}), \mathcal{C}) \simeq \text{SymMonCat}(\mathcal{A}, \mathcal{C}).$$

In particular, any symmetric monoidal functor $Q'(\mathcal{N}) \rightarrow \mathcal{C}$ (not just the strict ones, as we obtained in Display (5.4)) is isomorphic to one arising from a symmetric monoidal functor $\mathcal{A} \rightarrow \mathcal{C}$, giving full and faithful embeddings

$$(5.5) \quad [\mathcal{F}, \mathcal{C}] \hookrightarrow \text{SymMonCat}(\mathcal{A}, \mathcal{C})$$

$$(5.6) \quad \Gamma\text{-}\mathcal{C} = [\mathcal{F}, \mathcal{C}]_{\text{red}} \hookrightarrow \text{SymMonCat}(\mathcal{A}, \mathcal{C}).$$

We can always choose the symmetric monoidal equivalence $\mathcal{A} \rightarrow Q'(\mathcal{N})$ to be normal, in which case the final embedding actually becomes a full and faithful embedding

$$\Gamma\text{-}\mathcal{C} \hookrightarrow \text{SymMonCat}_{\text{nor}}(\mathcal{A}, \mathcal{C}).$$

Mandell [Man10] gives an explicit description of such a permutative category \mathcal{A} .

Definition 5.7. The objects of \mathcal{A} are (possibly empty) tuples $\vec{m} = (m_1, \dots, m_r)$, where each m_i is a positive integer. The morphism set

$$\mathcal{A}(\vec{m}, \vec{n}) \subset \text{Set}(\coprod_i \underline{m}_i, \coprod_j \underline{n}_j)$$

consists of those maps for which the inverse image of each \underline{n}_j is either empty or contained in a single \underline{m}_i . The monoidal product is given by concatenation, and the rest of the permutative structure is straightforward to spell out (see [Man10]).

We will need the following notation to describe maps in \mathcal{A} .

Notation 5.8. Suppose $\bar{\phi} : \vec{m} \rightarrow \vec{n}$ is a map in \mathcal{A} . For each i , let

$$(5.9) \quad \mathbf{p}(\bar{\phi}, i) = \{j \mid \emptyset \neq \bar{\phi}^{-1}(\underline{n}_j) \subset \underline{m}_i\}$$

and let

$$(5.10) \quad \phi_i = \bar{\phi}|_{\underline{m}_i} : \underline{m}_i \rightarrow \coprod_{j \in \mathbf{p}(\bar{\phi}, i)} \underline{n}_j.$$

For each $j \in \mathbf{p}(\bar{\phi}, i)$, let

$$\phi_{i,j} : \underline{m}_i \rightarrow \underline{n}_j$$

be the pointed map that restricts to ϕ_i on the elements in the preimage of \underline{n}_j and sends everything else to the basepoint.

Remark 5.11. Note that every morphism $\bar{\phi}$ in \mathcal{A} can be decomposed as

$$\coprod_i \underline{m}_i \rightarrow \coprod_i \coprod_{\mathbf{p}(\bar{\phi}, i)} \underline{n}_j \rightarrow \coprod_j \underline{n}_j$$

where the first map is given by $\coprod_i \phi_i$ and the second is given by reordering and inclusion of indexing sets. Each ϕ_i can be decomposed in \mathcal{A} as a partition of \underline{m}_i followed by the disjoint union of maps given by $\phi_i|_{\bar{\phi}^{-1}(\underline{n}_j)}$ and a reindexing:

$$\underline{m}_i \rightarrow \coprod_{j \in \mathbf{p}(\bar{\phi}, i)} |\phi_i^{-1}(\underline{n}_j)| \cong \coprod_{j \in \mathbf{p}(\bar{\phi}, i)} \phi_i^{-1}(\underline{n}_j) \rightarrow \coprod_{j \in \mathbf{p}(\bar{\phi}, i)} \underline{n}_j.$$

Definition 5.12. For $X \in \Gamma\text{-}\mathcal{C}$, the symmetric strict monoidal functor $AX : \mathcal{A} \rightarrow \mathcal{C}$ is defined as follows:

- For $\vec{m} = (m_1, \dots, m_r) \in \mathcal{A}$,

$$AX(\vec{m}) = \prod_i X(\underline{m}_i)$$

and $AX() = X(\underline{0}_+) = *$,

- For $\bar{\phi} \in \mathcal{A}(\vec{m}, \vec{n})$,

$$\bar{\phi}_* : X(\vec{m}) \rightarrow X(\vec{n})$$

is defined by the composite

$$AX(\vec{m}) \rightarrow \prod_i \prod_{j \in \mathbf{p}(\bar{\phi}, i)} X(\underline{n}_j) \xrightarrow{\tau} AX(\vec{n}).$$

The first map is given, for each i , by components $(\phi_{i,j})_*$. The second map is given by permuting factors and inserting the unique maps $X(\underline{0}_+) = * \rightarrow X(\underline{n}_j)$ for those \underline{n}_j such that $\bar{\phi}^{-1}(\underline{n}_j) = \emptyset$.

Note that by construction AX is a symmetric monoidal diagram with respect to concatenation in \mathcal{A} and the cartesian product in \mathcal{C} .

5.2. Extending A to Γ -lax maps. The assignment $X \mapsto AX$ is functorial in strict maps of diagrams, but we will need more.

Proposition 5.13. *The functor $A : \Gamma\text{-}2\mathcal{C}\text{at} \rightarrow \text{SymMonCat}(\mathcal{A}, 2\mathcal{C}\text{at})$ extends to a functor*

$$A : \Gamma\text{-}2\mathcal{C}\text{at}_l \rightarrow \mathcal{A}\text{-}2\mathcal{C}\text{at}_l.$$

Before beginning the proof, we define the relevant extension of A .

Definition 5.14. Let X and Y be Γ -2-categories, and let $h : X \rightarrow Y$ be a Γ -lax map. We define an \mathcal{A} -lax map $Ah : AX \rightarrow AY$. Unpacking Definition 4.2 we see that such a map consists of, for each object \vec{m} of \mathcal{A} , a 2-functor

$$AX(\vec{m}) \rightarrow AY(\vec{m})$$

and for each morphism $\bar{\phi} : \vec{m} \rightarrow \vec{n}$ in \mathcal{A} , a 2-natural transformation

$$h_{\bar{\phi}} : \bar{\phi}_* h_{\vec{m}} \Rightarrow h_{\vec{n}} \bar{\phi}_*.$$

The 2-functor $(Ah)_{\vec{m}}$ is given by

$$AX(\vec{m}) = \prod_i X(\underline{m}_{i+}) \xrightarrow{\prod_i h_{m_{i+}}} \prod_i Y(\underline{m}_{i+}) = AY(\vec{m}).$$

We define the 2-natural transformation $(Ah)_{\bar{\phi}}$ as follows: For each $j \in \mathfrak{p}(\bar{\phi}, i)$, $\phi_{i,j}$ is a map in \mathcal{F} and therefore since h is Γ -lax we have 2-natural transformations

$$h_{\phi_{i,j}} : \phi_{i,j} h_{\underline{m}_{i+}} \Rightarrow h_{\underline{n}_{j+}} \phi_{i,j}.$$

Taking the product over $j \in \mathfrak{p} = \mathfrak{p}(\bar{\phi}, i)$ defines 2-natural transformations h_{ϕ_i} :

$$(5.15) \quad h_{\phi_i} : \phi_i h_{\underline{m}_{i+}} \Rightarrow \left(\prod_{j \in \mathfrak{p}} h_{\underline{n}_{j+}} \right) \phi_i.$$

Taking the product over i gives the top square of the diagram below.

$$(5.16) \quad \begin{array}{ccc} AX(\vec{m}) & \xrightarrow{h_{\vec{m}}} & AY(\vec{m}) \\ \downarrow \Pi_i \phi_i & \nearrow \Pi_i h_{\phi_i} & \downarrow \Pi_i \phi_i \\ \prod_i \prod_{j \in J} X(\underline{n}_{j+}) & \xrightarrow{\prod_i \prod_{j \in \mathfrak{p}} h_{\underline{n}_{j+}}} & \prod_i \prod_{j \in J} Y(\underline{n}_{j+}) \\ \downarrow \tau & = & \downarrow \tau \\ AX(\vec{n}) & \xrightarrow{h_{\vec{n}}} & AY(\vec{n}) \end{array}$$

In the bottom square the maps τ are given by permuting factors in the cartesian product and hence the bottom square commutes. We define $Ah_{\bar{\phi}}$ as the pasting of these two cells. The 2-naturality of $Ah_{\bar{\phi}}$ is immediate by the 2-naturality of each $h_{\phi_{i,j}}$ and τ .

Proof of Proposition 5.13. We verify that the extension of A given in Definition 5.14 defines a functor as claimed. This consists of verifying the following equality of pasting diagrams for Γ -lax maps

$$X \xrightarrow{h} Y \xrightarrow{k} Z.$$

$$\begin{array}{ccccc}
AX(\vec{m}) & \xrightarrow{h_{\vec{m}}} & AY(\vec{m}) & \xrightarrow{k_{\vec{m}}} & AZ(\vec{m}) \\
\downarrow \Pi_i \phi_i & \swarrow \Pi_i h_{\phi_i} & \downarrow \Pi_i \phi_i & \swarrow \Pi_i k_{\phi_i} & \downarrow \Pi_i \phi_i \\
\Pi_i \prod_{j \in J} X(\underline{n_j}_+) & \xrightarrow{\Pi_i \prod_{j \in \mathbb{P}} h_{\underline{n_j}_+}} & \Pi_i \prod_{j \in J} Y(\underline{n_j}_+) & \xrightarrow{\Pi_i \prod_{j \in \mathbb{P}} k_{\underline{n_j}_+}} & \Pi_i \prod_{j \in J} Z(\underline{n_j}_+) \\
& & \parallel & & \\
AX(\vec{m}) & \xrightarrow{(kh)_{\vec{m}}} & & & AZ(\vec{m}) \\
\downarrow \Pi_i \phi_i & \swarrow \Pi_i (kh)_{\phi_i} & & & \downarrow \Pi_i \phi_i \\
\Pi_i \prod_{j \in J} X(\underline{n_j}_+) & \xrightarrow{\Pi_i \prod_{j \in \mathbb{P}} (kh)_{\underline{n_j}_+}} & & & \Pi_i \prod_{j \in J} Z(\underline{n_j}_+)
\end{array}$$

But this equality is immediate because composition is strict in the (1-)category $\Gamma\text{-}2\mathcal{C}at_l$ (see Remark 4.4). \square

Recalling the notions of symmetric monoidal diagram and monoidal lax map from Section 4.4, the following is immediate from the definition of A .

Proposition 5.17. *Let X and Y be Γ -2-categories and $h: X \rightarrow Y$ be a Γ -lax map. Then the \mathcal{A} -lax map $Ah: AX \rightarrow AY$ is monoidal.*

5.3. Definition of P . Composing the functor $A: \Gamma\text{-}2\mathcal{C}at_l \rightarrow \mathcal{A}\text{-}2\mathcal{C}at_l$ of Proposition 5.13 with the Grothendieck construction of Section 4.3, we obtain a functor

$$P = \mathcal{A}\text{-}\int(A-): \Gamma\text{-}2\mathcal{C}at_l \rightarrow 2\mathcal{C}at.$$

Applying Propositions 4.31 and 5.17, we have the following refinement.

Theorem 5.18. *The assignment on objects $PX = \mathcal{A}\text{-}\int AX$ extends to a functor*

$$P: \Gamma\text{-}2\mathcal{C}at_l \rightarrow \text{Perm2Cat}$$

from the category of strict Γ -2-categories and Γ -lax maps to the category of permutative 2-categories and strict (2-)functors.

In the remainder of this section we give an explicit description of the objects, 1-cells, and 2-cells of PX , and then prove some basic homotopical results about P . The objects of PX are pairs $[\vec{m}, \vec{x}]$ for $\vec{m} = (m_i) \in \mathcal{A}$ and $\vec{x} = (x_i) \in AX(\vec{m})$. The 1- and 2-cells in PX are pairs as below:

$$\begin{array}{ccc}
& [\bar{\phi}, \vec{f}] & \\
[\vec{m}, \vec{x}] & \begin{array}{c} \swarrow \bar{\phi}, \vec{a} \\ \searrow \bar{\phi}, \vec{g} \end{array} & [\vec{n}, \vec{y}] \\
& [\bar{\phi}, \vec{g}] &
\end{array}$$

where $\bar{\phi}: \vec{m} \rightarrow \vec{n}$ is a map in \mathcal{A} and $\vec{f}, \vec{g}, \vec{a}$ are cells in $AX(\vec{n})$:

$$\begin{array}{ccc}
& \vec{f} & \\
\bar{\phi}_* \vec{x} & \begin{array}{c} \swarrow \bar{\phi}, \vec{a} \\ \searrow \vec{g} \end{array} & \vec{y} \\
& \vec{g} &
\end{array}$$

Remark 5.19. The definition of composition in PX immediately yields the following special cases for $\bar{\phi}: \vec{m} \rightarrow \vec{n}$ in \mathcal{A} and appropriately composable 1-cells \vec{f} in $AX(\vec{m})$ and \vec{g} in $AX(\vec{n})$:

$$[\text{id}, \vec{g}][\bar{\phi}, \text{id}] = [\bar{\phi}, \vec{g}],$$

$$[\bar{\phi}, \vec{g}][\text{id}, \vec{f}] = [\bar{\phi}, \vec{g} \circ \bar{\phi}_* \vec{f}].$$

Remark 5.20. For a Γ -lax map $h: X \rightarrow Y$, the functor $Ph: PX \rightarrow PY$ is given explicitly as:

On 0-cells

$$Ph[\vec{m}, \vec{x}] = [\vec{m}, Ah_{\vec{m}}(\vec{x})].$$

On 1-cells

$$Ph[\bar{\phi}, \vec{f}] = [\bar{\phi}, Ah_{\vec{n}}(\vec{f}) \circ Ah_{\bar{\phi}}].$$

On 2-cells

$$Ph[\bar{\phi}, \vec{a}] = [\bar{\phi}, Ah_{\vec{n}}(\vec{a}) * 1_{Ah_{\bar{\phi}}}].$$

These are given by the following 0-, 1-, and 2-cells in AY .

$$\begin{array}{ccccc} & & Ah_{\vec{n}}(\vec{f}) & & \\ & \xrightarrow{Ah_{\bar{\phi}}} & Ah_{\vec{n}}(\bar{\phi}_* \vec{x}) & \begin{array}{c} \nearrow Ah_{\vec{n}}(\vec{f}) \\ \Downarrow Ah_{\vec{n}}(\vec{a}) \\ \searrow Ah_{\vec{n}}(\vec{g}) \end{array} & \\ \bar{\phi}_* Ah_{\vec{m}}(\vec{x}) & & Ah_{\vec{n}}(\bar{\phi}_* \vec{x}) & & Ah_{\vec{n}}(\vec{y}) \end{array}$$

The following two results make use of the equivalence

$$(5.21) \quad \text{hocolim}_{\mathcal{A}} N(AX) \xrightarrow{\sim} N(\mathcal{A} \int AX)$$

due to Thomason for diagrams of categories and generalized to diagrams of bicategories by Carrasco-Cegarra-Garzón [CCG10, Theorem 7.3].

Lemma 5.22. *Let X be a special Γ -2-category. Then the inclusion*

$$X(\underline{1}_+) \rightarrow PX$$

is a weak equivalence of 2-categories.

Proof. We have the following chain of weak equivalences

$$X(\underline{1}_+) \xrightarrow{\sim} \text{hocolim}_{\mathcal{A}} N(X) \xrightarrow{\sim} \text{hocolim}_{\mathcal{A}} N(AX) \xrightarrow{\sim} N(\mathcal{A} \int AX) = N(PX).$$

The last equivalence is Display (5.21) and the others follow from changing indexing categories as in the argument of [Man10, Theorem 5.3]. \square

Proposition 5.23. *The functor $P: \Gamma\text{-}2\text{Cat}_l \rightarrow \text{Perm2Cat}$ sends levelwise weak equivalences of Γ -2-categories to weak equivalences of permutative 2-categories and therefore is a relative functor*

$$(\Gamma\text{-}2\text{Cat}_l, \mathcal{W}) \rightarrow (\text{Perm2Cat}, \mathcal{W}).$$

Proof. This is immediate from Display (5.21). \square

6. K-THEORY CONSTRUCTIONS

In this section we give two constructions of K -theory for 2-categories. The first, in Section 6.1, is a construction for permutative Gray-monoids and has a counit described in Section 6.2. The second, in Section 6.3, is a construction for permutative 2-categories, taking advantage of their additional strictness. Using both of these, we prove in Theorem 6.44 that the homotopy theory of permutative Gray-monoids is equivalent to that of permutative 2-categories. The stricter construction is also essential for the triangle identities of Sections 7.5 and 7.6.

6.1. K -theory for permutative Gray-monoids. Recall that $\mathcal{P}erm\mathcal{G}ray\mathcal{M}on_{\text{nop}}$ denotes the category of permutative Gray-monoids and normal, oplax functors between them. We define a functor

$$\tilde{K}: \mathcal{P}erm\mathcal{G}ray\mathcal{M}on_{\text{nop}} \rightarrow \Gamma\text{-}2\mathcal{C}at$$

using a construction very similar to that of [Oso12, §5.2].

Let $(\mathcal{C}, \oplus, e, \beta)$ be a permutative Gray-monoid. The Γ -2-category $\tilde{K}\mathcal{C}$ is given in Constructions 6.1 and 6.8. In Proposition 6.9 we show that \tilde{K} is functorial with respect to normal oplax functors of permutative Gray-monoids, and in Proposition 6.12 we show that $\tilde{K}\mathcal{C}$ is a special Γ -2-category.

Construction 6.1. Let $\underline{n}_+ \in \mathcal{F}$ be a finite pointed set. We define a 2-category $\tilde{K}\mathcal{C}(\underline{n}_+)$ as follows:

- Objects are given by collections of the form $\{x_s, c_{s,t}\}$ where x_s is an object of \mathcal{C} for each $s \in \underline{n}$ and $c_{s,t}: x_{s \cup t} \rightarrow x_s \oplus x_t$ is a 1-cell of \mathcal{C} for each such pair (s, t) with $s \cap t = \emptyset$. These collections are required to satisfy the following axioms:
 - $x_\emptyset = e$;
 - $c_{\emptyset, s} = \text{id}_{x_s} = c_{s, \emptyset}$;
 - for every triple (s, t, u) of pairwise disjoint subsets the diagram below commutes;

$$(6.2) \quad \begin{array}{ccc} x_{s \cup t \cup u} & \xrightarrow{c_{s, t \cup u}} & x_s \oplus x_{t \cup u} \\ c_{s \cup t, u} \downarrow & & \downarrow \text{id} \oplus c_{t, u} \\ x_{s \cup t} \oplus x_u & \xrightarrow{c_{s, t} \oplus \text{id}} & x_s \oplus x_t \oplus x_u \end{array}$$

- for every pair of disjoint subsets s, t , the diagram below commutes.

$$(6.3) \quad \begin{array}{ccc} x_{s \cup t} & \xrightarrow{c_{s, t}} & x_s \oplus x_t \\ \parallel & & \downarrow \beta_{x_s, x_t} \\ x_{t \cup s} & \xrightarrow{c_{t, s}} & x_t \oplus x_s \end{array}$$

- A 1-morphism from $\{x_s, c_{s,t}\}$ to $\{x'_s, c'_{s,t}\}$ is a collection $\{f_s, \gamma_{s,t}\}$, where s, t are as above; $f_s: x_s \rightarrow x'_s$ is a 1-morphism in \mathcal{C} and $\gamma_{s,t}$ is an invertible 2-cell:

$$\begin{array}{ccc} x_{s \cup t} & \xrightarrow{c_{s, t}} & x_s \oplus x_t \\ f_{s \cup t} \downarrow & \gamma_{s, t} \swarrow & \downarrow f_s \oplus \text{id} \\ x'_{s \cup t} & \xrightarrow{c'_{s, t}} & x'_s \oplus x'_t \\ & & \downarrow \text{id} \oplus f_t \end{array}$$

These are required to satisfy the following axioms:

- $f_\emptyset = \text{id}_e$;
- $\gamma_{\emptyset, s}$ is the identity 2-morphism

$$(f_s \oplus \text{id}_e) \circ \text{id}_{x_s} = f_s \implies f_s = \text{id}_{x'_s} \circ f_s$$

and similarly for $\gamma_{s,\emptyset}$;

c) for every triple of pairwise disjoint subsets s, t, u we have the following equality of pasting diagrams:

$$\begin{array}{ccccc}
 x_{s \cup t \cup u} & \xrightarrow{c_{s,t \cup u}} & x_s \oplus x_{t \cup u} & \xrightarrow{\text{id} \oplus c_{t,u}} & x_s \oplus x_t \oplus x_u \\
 \downarrow f_{s \cup t \cup u} & \searrow \gamma_{s,t \cup u} & \downarrow f_s \oplus \text{id} & \searrow \Sigma & \downarrow f_s \oplus \text{id} \\
 x'_s \oplus x_{t \cup u} & \xrightarrow{\text{id} \oplus c_{t,u}} & x'_s \oplus x_t \oplus x_u & & \\
 \downarrow \text{id} \oplus f_{t \cup u} & & \downarrow \text{id} \oplus f_t \oplus \text{id} & & \\
 x'_s \oplus x'_t \oplus x_u & & & & \downarrow \text{id} \oplus f_u \\
 \downarrow f_{s \cup t \cup u} & & & & \\
 x'_{s \cup t \cup u} & \xrightarrow{c'_{s,t \cup u}} & x'_s \oplus x'_{t \cup u} & \xrightarrow{\text{id} \oplus c'_{t,u}} & x'_s \oplus x'_t \oplus x'_u
 \end{array}
 =
 \begin{array}{ccccc}
 x_{s \cup t \cup u} & \xrightarrow{c_{s \cup t,u}} & x_{s \cup t} \oplus x_u & \xrightarrow{c_{s,t} \oplus \text{id}} & x_s \oplus x_t \oplus x_u \\
 \downarrow f_{s \cup t \cup u} & \searrow \gamma_{s \cup t,u} & \downarrow f_{s \cup t} \oplus \text{id} & \searrow \gamma_{s,t} \oplus \text{id} & \downarrow f_s \oplus \text{id} \\
 x'_{s \cup t} \oplus x_u & \xrightarrow{c'_{s,t} \oplus \text{id}} & x'_s \oplus x'_t \oplus x_u & & \\
 \downarrow \text{id} \oplus f_u & & \downarrow \text{id} \oplus f_t \oplus \text{id} & & \\
 x'_{s \cup t \cup u} & \xrightarrow{c'_{s \cup t,u}} & x'_{s \cup t} \oplus x'_u & \xrightarrow{c'_{s,t} \oplus \text{id}} & x'_s \oplus x'_t \oplus x'_u
 \end{array}
 \quad (6.4)$$

d) for every pair of disjoint subsets s, t we have the following equality of pasting diagrams:

$$(6.5) \quad \begin{array}{ccccc} x_{s \cup t} & \xrightarrow{c_{s,t}} & x_s \oplus x_t & \xrightarrow{\beta} & x_t \oplus x_s \\ \downarrow f_{s \cup t} & \nearrow \gamma_{s,t} & \downarrow f_s \oplus \text{id} & = & \text{id} \oplus f_s \swarrow \quad \swarrow f_t \oplus \text{id} \\ x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x_t & \xrightarrow{\beta} & x'_t \oplus x_s \\ & & \downarrow \text{id} \oplus f_t & = & \downarrow f_t \oplus \text{id} \quad \downarrow \text{id} \oplus f_s \\ x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x'_t & \xrightarrow{\beta} & x'_t \oplus x'_s \\ & & & = & \\ & & x_{t \cup s} & \xrightarrow{c_{t,s}} & x_t \oplus x_s \\ \downarrow f_{t \cup s} & \nearrow \gamma_{t,s} & \downarrow f_t \oplus \text{id} & & \downarrow \text{id} \oplus f_s \\ x'_{t \cup s} & \xrightarrow{c'_{t,s}} & x'_t \oplus x_s & & \end{array}$$

iii. For 1-morphisms $\{f_s, \gamma_{s,t}\}, \{g_s, \delta_{s,t}\} : \{x_s, c_{s,t}\} \rightarrow \{x'_s, c'_{s,t}\}$, a 2-morphism between them is a collection $\{\alpha_s\}$ of 2-morphisms in \mathcal{C} , $\alpha_s : f_s \Rightarrow g_s$, such that for all s, t as above we have the following equality of pasting diagrams:

$$(6.6) \quad \begin{array}{ccc} x_{s \cup t} & \xrightarrow{c_{s,t}} & x_s \oplus x_t \\ \downarrow g_{s \cup t} & \nearrow \alpha_{s \cup t} & \downarrow f_{s \cup t} \quad \nearrow \gamma_{s,t} \\ x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x_t \\ & & \downarrow \text{id} \oplus f_t \\ & & x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x'_t \\ & & & & \\ x_{s \cup t} & \xrightarrow{c_{s,t}} & x_s \oplus x_t & & x_{s \cup t} & \xrightarrow{c_{s,t}} & x_s \oplus x_t \\ \downarrow g_{s \cup t} & \nearrow \delta_{s,t} & \downarrow f_s \oplus \text{id} & & \downarrow g_{s \cup t} & \nearrow \alpha_s \oplus 1 & \downarrow f_s \oplus \text{id} \\ x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x_t & & x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x_t \\ & & \downarrow \text{id} \oplus g_t & & & & \downarrow \text{id} \oplus g_t \\ & & x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x'_t & & x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x'_t \end{array}$$

Vertical composition of 2-morphisms is defined componentwise. Horizontal composition of 1-cells and 2-cells given by:

$$\begin{aligned} \{g_s, \delta_{s,t}\} \circ \{f_s, \gamma_{s,t}\} &= \{g_s \circ f_s, (\delta \diamond \gamma)_{s,t}\} \\ \{\alpha'_s\} * \{\alpha_s\} &= \{\alpha'_s * \alpha_s\}, \end{aligned}$$

where the 2-morphism $(\delta \diamond \gamma)_{s,t}$ is defined by the pasting diagram below.

$$\begin{array}{ccccc}
 x_{s \cup t} & \xrightarrow{c_{s,t}} & x_s \oplus x_t & & \\
 \downarrow f_{s \cup t} & \nearrow \gamma_{s,t} & \downarrow f_s \oplus \text{id} & \nearrow (g_s f_s) \oplus \text{id} & \\
 x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x'_t & \xleftarrow{\Sigma} & x''_s \oplus x_t \\
 \downarrow g_{s \cup t} & \nearrow \delta_{s,t} & \downarrow g_s \oplus \text{id} & \nearrow \text{id} \oplus f_t & \downarrow \text{id} \oplus (g_t f_t) \\
 x''_{s \cup t} & \xrightarrow{c''_{s,t}} & x''_s \oplus x'_t & \xleftarrow{\Sigma} & x'_s \oplus x'_t
 \end{array}$$

Note that the axioms imply that $\tilde{K}\mathcal{C}(\underline{0}_+)$ has a unique object given by e , and only identity 1- and 2-cells.

Remark 6.7. In Section 7 we will use the fact that the 0-cells of $\tilde{K}\mathcal{C}(\underline{n}_+)$ can be thought of as functions on subsets and disjoint pairs of subsets of \underline{n} :

$$\left\{ \begin{array}{l} (s \subset \underline{n}) \mapsto x_s \\ (s, t \subset \underline{n}) \xrightarrow{s \cap t = \emptyset} c_{s,t} \end{array} \right\}.$$

The 1- and 2-cells can be thought of similarly.

Construction 6.8. Let $\phi: \underline{m}_+ \rightarrow \underline{n}_+$ be a map in \mathcal{F} . We define a strict 2-functor

$$\phi_*: \tilde{K}\mathcal{C}(\underline{m}_+) \rightarrow \tilde{K}\mathcal{C}(\underline{n}_+)$$

as follows

$$\begin{aligned}
 \phi_*\{x_s, c_{s,t}\} &= \{x_u^\phi, c_{u,v}^\phi\} = \{x_{\phi^{-1}(u)}, c_{\phi^{-1}(u), \phi^{-1}(v)}\} \\
 \phi_*\{f_s, \gamma_{s,t}\} &= \{f_u^\phi, \gamma_{u,v}^\phi\} = \{f_{\phi^{-1}(u)}, \gamma_{\phi^{-1}(u), \phi^{-1}(v)}\} \\
 \phi_*\{\alpha_s\} &= \{\alpha_u^\phi\} = \{\alpha_{\phi^{-1}(u)}\},
 \end{aligned}$$

where s, t range over disjoint subsets of \underline{m} and u, v range over disjoint subsets of \underline{n} . Since ϕ is basepoint preserving, $\phi^{-1}(u)$ does not contain the basepoint and it is an allowed indexing subset of \underline{m} . Since u and v are disjoint, their preimages under ϕ are also disjoint. Note that if $\psi: \underline{n}_+ \rightarrow \underline{p}_+$ is another map in \mathcal{F} then we have $(\psi\phi)_* = \psi_*\phi_*$, so $\tilde{K}\mathcal{C}$ is a Γ -2-category.

Proposition 6.9. *The construction above gives the object assignment for a functor*

$$\tilde{K}: \text{PermGrayMon}_{\text{nop}} \rightarrow \Gamma\text{-}2\text{Cat}.$$

Proof. Let (F, θ) be a normal oplax functor between the permutative Gray-monoids \mathcal{C} and \mathcal{D} . We first define a 2-functor $\tilde{K}F_{\underline{n}_+}: \tilde{K}\mathcal{C}(\underline{n}_+) \rightarrow \tilde{K}\mathcal{D}(\underline{n}_+)$. This is given for objects,

1-cells, and 2-cells as:

$$\begin{aligned}\{x_s, c_{s,t}\} &\longmapsto \{F(x_s), \theta \circ F(c_{s,t})\} \\ \{f_s, \gamma_{s,t}\} &\longmapsto \{F(f_s), 1_\theta * F(\gamma_{s,t})\} \\ \{\alpha_s\} &\longmapsto \{F(\alpha_s)\},\end{aligned}$$

where $1_\theta * F(\gamma_{s,t})$ is the 2-cell defined by the pasting diagram below.

$$\begin{array}{ccccc} F(x_{s \cup t}) & \xrightarrow{F(c_{s,t})} & F(x_s \oplus x_t) & \xrightarrow{\theta} & F(x_s) \oplus F(x_t) \\ & \downarrow & \downarrow F(f_s \oplus \text{id}) & = & \downarrow F(f_s) \oplus \text{id} \\ F(f_{s \cup t}) & \swarrow F(\gamma_{s,t}) & F(x'_s \oplus x_t) & \xrightarrow{\theta} & F(x'_s) \oplus F(x_t) \\ & \downarrow & \downarrow F(\text{id} \oplus f_t) & = & \downarrow \text{id} \oplus F(f_t) \\ F(x'_{s \cup t}) & \xrightarrow{F(c'_{s,t})} & F(x'_s \oplus x'_t) & \xrightarrow{\theta} & F(x'_s) \oplus F(x'_t). \end{array}$$

Now one must verify that these assignments send k -cells to k -cells for $k = 0, 1, 2$, and then check 2-functoriality. This is largely routine using the permutative Gray-monoid axioms (including axioms for Gray tensor product and its interaction with β) and normal oplax functor axioms. As an example, we include the verification of the axiom in Display (6.2) as part of checking that $\{F(x_s), \theta \circ F(c_{s,t})\}$ is an object of $\tilde{K}\mathcal{D}(\underline{n}_+)$. The other verifications are similarly straightforward. We must verify that the following diagram of 1-morphisms in \mathcal{D} commutes.

$$\begin{array}{ccccc} F(x_{s \cup t \cup u}) & \xrightarrow{F(c_{s,t \cup u})} & F(x_s \oplus x_{t \cup u}) & \xrightarrow{\theta} & F(x_s) \oplus F(x_{t \cup u}) \\ \downarrow F(c_{s,t \cup u}) & & \downarrow F(\text{id} \oplus c_{t,u}) & & \downarrow \text{id} \oplus F(c_{t,u}) \\ F(x_{s \cup t} \oplus x_u) & \xrightarrow{F(c_{s,t} \oplus \text{id})} & F(x_s \oplus x_t \oplus x_u) & \xrightarrow{\theta} & F(x_s) \oplus F(x_t \oplus x_u) \\ \downarrow \theta & & \downarrow \theta & & \downarrow \text{id} \oplus \theta \\ F(x_{s \cup t}) \oplus F(x_u) & \xrightarrow{F(c_{s,t}) \oplus \text{id}} & F(x_s \oplus x_t) \oplus F(x_u) & \xrightarrow{\theta \oplus \text{id}} & F(x_s) \oplus F(x_t) \oplus F(x_u) \end{array}$$

The upper left square commutes because F is a 2-functor, and the diagram commutes before applying F . The upper right and lower left squares commute since θ is a 2-natural transformation. Finally, the commutativity of the lower right square is one of the axioms for normal oplax functors.

Now the collection of 2-functors $\tilde{K}\mathcal{F}_{\underline{n}_+}$ is natural with respect to \underline{n}_+ , and therefore $\tilde{K}F$ is a strict map of Γ -2-categories. Finally, it is easy to check that \tilde{K} is functorial. \square

Remark 6.10. Let \mathcal{C} be a permutative Gray-monoid, $\underline{m}_+ \in \mathcal{F}$, and $\{x, c\} \in \tilde{K}\mathcal{C}(\underline{m}_+)$. Given a subset $s \subset \underline{m}$ and a partition $s = s_1 \cup \dots \cup s_a$, there are, a priori, many 1-cells

$$x_s \rightarrow x_{s_1} \oplus \dots \oplus x_{s_a}$$

in $\tilde{K}\mathcal{C}(\underline{m}_+)$ given by composing instances of c and β . However conditions (6.2) and (6.3), together with 2-naturality of β with respect to maps in the Gray tensor product, ensure that these are all equal.

Likewise, given $\{f, \gamma\}: \{x, c\} \rightarrow \{y, d\}$, all 2-cells

$$\begin{array}{ccc} x_s & \xrightarrow{!} & \oplus_i x_{s_i} \\ f_s \downarrow & \nearrow \gamma & \downarrow \oplus_i f_{s_i} \\ y_s & \xrightarrow{!} & \oplus_i y_{s_i} \end{array}$$

given by pasting instances of γ and β are equal. Recall that $\oplus_i f_{s_i}$ is defined in Notation 3.24.

Definition 6.11. Let \mathcal{C}, \mathcal{D} be a pair of permutative Gray-monoids. A strict functor of permutative Gray-monoids $F: \mathcal{C} \rightarrow \mathcal{D}$ is a *stable equivalence* if $\tilde{K}F$ is a stable equivalence of Γ -2-categories. We let $(\text{PermGrayMon}, \mathcal{S})$ denote the relative category of permutative Gray-monoids with stable equivalences.

The next proposition follows from [Oso12, §5.2] and has Proposition 6.13 as an immediate consequence. In Proposition 6.13 we use the fact that every strict functor is a normal oplax one and hence implicitly restrict \tilde{K} to the subcategory PermGrayMon .

Proposition 6.12. *The Γ -2-category $\tilde{K}\mathcal{C}$ is special, with $\tilde{K}\mathcal{C}(1_+)$ isomorphic to \mathcal{C} .*

Proposition 6.13. *The functor $\tilde{K}: \text{PermGrayMon} \rightarrow \Gamma\text{-2Cat}$ preserves weak equivalences and is therefore a relative functor $\tilde{K}: (\text{PermGrayMon}, \mathcal{W}) \rightarrow (\Gamma\text{-2Cat}, \mathcal{W})$.*

Remark 6.14. In [GO13], the authors use their coherence theorem to construct a K -theory functor for all symmetric monoidal bicategories, by first constructing a pseudo-diagram of bicategories indexed on \mathcal{F} , and then rectifying it using the methods of [CCG11]. When the input is a permutative Gray-monoid, one can use the same technique as in [GO13, §2] to prove that the two resulting Γ -bicategories are levelwise weakly equivalent. We have chosen to use this explicit construction because of its functoriality.

6.2. Counit for permutative Gray-monoids. Let \mathcal{C} be a permutative Gray-monoid. We now use (weak) functoriality of the Grothendieck construction [CCG11, §3.2] to give a symmetric monoidal pseudofunctor

$$\tilde{\varepsilon}: P\tilde{K}\mathcal{C} \rightarrow \mathcal{C}.$$

Recalling Definition 3.3, this requires that we construct pseudofunctors

$$\tilde{\varepsilon}_{\vec{m}}: A\tilde{K}\mathcal{C}(\vec{m}) \rightarrow \mathcal{C}$$

for each object \vec{m} in \mathcal{A} , together with pseudonatural transformations

$$\begin{array}{ccc} A\tilde{K}\mathcal{C}(\vec{m}) & & \\ \downarrow \tilde{\varepsilon}_{\vec{m}} & \nearrow \tilde{\varepsilon}_{\vec{n}} & \\ A\tilde{K}\mathcal{C}(\vec{n}) & & \end{array}$$

for each morphism $\bar{\phi}: \vec{m} \rightarrow \vec{n}$ in \mathcal{A} . For the general situation considered in [CCG11], there are further modifications, but we will verify that these are in fact identities. Applying the Grothendieck construction therefore produces $\tilde{\varepsilon}$ as a pseudofunctor between bicategories, the source and target of which happen to be 2-categories.

Lemma 6.15. *Let \mathcal{C} be a permutative Gray-monoid. Evaluation at a subset $s \subset \underline{m}$ is a 2-functor*

$$\text{ev}_s: \tilde{K}\mathcal{C}(\underline{m}_+) \rightarrow \mathcal{C}.$$

Definition 6.16. Let \mathcal{C} be a permutative Gray-monoid, and let $\vec{m} \in \mathcal{A}$. Then $\tilde{\varepsilon}_{\vec{m}}$ is defined to be the composite

$$A\tilde{K}\mathcal{C}(\vec{m}) = \prod_i \tilde{K}\mathcal{C}(\underline{m}_i) \xrightarrow{\prod_i \text{ev}_{m_i}} \prod_i \mathcal{C} \xrightarrow{\oplus} \mathcal{C}.$$

Since \oplus is cubical and therefore only a pseudofunctor and each ev_{m_i} is a 2-functor, the composite $\tilde{\varepsilon}_{\vec{m}}$ is a pseudofunctor. For the empty sequence (\cdot) , $\tilde{\varepsilon}_{(\cdot)}$ is defined as the 2-functor $* \rightarrow \mathcal{C}$ that sends the point to the unit object e of the monoidal structure.

Notation 6.17. For each i , recall the following notation of Display (5.9):

$$\mathfrak{p}(\bar{\phi}, i) = \{j \mid \emptyset \neq \bar{\phi}^{-1}(\underline{n}_j) \subset \underline{m}_i\}.$$

The restriction of $\bar{\phi}$ to \underline{m}_i gives a partition of \underline{m}_i into

$$\coprod_{j \in \mathfrak{p}(\bar{\phi}, i)} \bar{\phi}^{-1}(\underline{n}_j),$$

where we order $\mathfrak{p}(\bar{\phi}, i)$ as a subset of the indexing of the tuple \vec{n} .

For an object $\overrightarrow{\{x, c\}} = \{\vec{x}, \vec{c}\}$ of $\tilde{K}\mathcal{C}(\vec{m})$, let

$$(6.18) \quad c_{\bar{\phi}}^i: x_{\underline{m}_i}^i \rightarrow \bigoplus_{j \in \mathfrak{p}(\bar{\phi}, i)} x_{\bar{\phi}^{-1}(\underline{n}_j)}^i$$

be the unique 1-cell determined by c^i and the partition above (see Remark 6.10). Using the symmetry of \mathcal{C} to reorder, we obtain each 1-cell $\tilde{\varepsilon}_{\bar{\phi}}(\{\vec{x}, \vec{c}\})$ as the composite:

$$(6.19) \quad \bigoplus_i x_{\underline{m}_i}^i \xrightarrow{\oplus_i c_{\bar{\phi}}^i} \bigoplus_i \bigoplus_{j \in \mathfrak{p}(\bar{\phi}, i)} x_{\bar{\phi}^{-1}(\underline{n}_j)}^i \xrightarrow{\beta} \bigoplus_j (\tilde{\varepsilon}_{\vec{n}}(\bar{\phi}_* \{\vec{x}, \vec{c}\}))^j$$

where

$$(\tilde{\varepsilon}_{\vec{n}}(\bar{\phi}_* \{\vec{x}, \vec{c}\}))^j = x_{\bar{\phi}^{-1}(\underline{n}_j)}^{i_j}$$

for $\bar{\phi}^{-1}(\underline{n}_j) \subset \underline{m}_{i_j}$. Note that this is well-defined because i_j is uniquely determined if $\bar{\phi}^{-1}(\underline{n}_j) \neq \emptyset$ and $x_{\bar{\phi}}^i = e$ for any i .

For a morphism in $A\tilde{K}\mathcal{C}(\vec{m})$

$$\overrightarrow{\{f, \gamma\}} = \{\vec{f}, \vec{\gamma}\}: \{\vec{x}, \vec{c}\} \rightarrow \{\vec{y}, \vec{d}\}$$

we have a 2-cell given by the pasting

(6.20)

$$\begin{array}{ccccc} \tilde{\varepsilon}_{\vec{m}}(\{\vec{x}, \vec{c}\}) & \xrightarrow{\oplus_i c_{\bar{\phi}}^i} & \bigoplus_i \bigoplus_{j \in \mathfrak{p}(\bar{\phi}, i)} x_{\bar{\phi}^{-1}(\underline{n}_j)}^i & \xrightarrow{\beta} & \tilde{\varepsilon}_{\vec{n}}(\bar{\phi}_* \{\vec{x}, \vec{c}\}) \\ \tilde{\varepsilon}_{\vec{m}}(\{\vec{f}, \vec{\gamma}\}) \downarrow & \swarrow & \downarrow \oplus_i \oplus_j f^i & \swarrow & \downarrow \tilde{\varepsilon}_{\vec{n}}(\bar{\phi}_* \{\vec{f}, \vec{\gamma}\}) \\ \tilde{\varepsilon}_{\vec{m}}(\{\vec{y}, \vec{d}\}) & \xrightarrow{\oplus_i d_{\bar{\phi}}^i} & \bigoplus_i \bigoplus_{j \in \mathfrak{p}(\bar{\phi}, i)} y_{\bar{\phi}^{-1}(\underline{n}_j)}^i & \xrightarrow{\beta} & \tilde{\varepsilon}_{\vec{n}}(\bar{\phi}_* \{\vec{y}, \vec{d}\}) \end{array}$$

where the left-hand 2-cell is given as in Remark 6.10 and the right-hand 2-cell is given by pseudonaturality of β .

Lemma 6.21. *The above data constitute a pseudonatural transformation $\tilde{\varepsilon}_{\bar{\phi}}$.*

Remark 6.22. When $\vec{m} = (m)$ and $\vec{n} = (n)$ are tuples of length one and $\bar{\phi}$ consists of a single map of finite sets

$$\underline{m} \rightarrow \underline{n},$$

then $\tilde{\varepsilon}_{\bar{\phi}}$ is the identity pseudo natural transformation.

Remark 6.23. For general \vec{m}, \vec{n} , if $\bar{\phi}$ is a block permutation of the \underline{m}_i but does not partition or permute the elements of any \underline{m}_i , then the formulas in Display (6.19) and Construction 6.8 show that $\tilde{\varepsilon}_{\bar{\phi}}$ is given by the component of β for the corresponding permutation of summands $x_{\underline{m}_i}^i$. In particular, for the map $\text{id}: \vec{m} \rightarrow \vec{m}$, the pseudonatural transformation $\tilde{\varepsilon}_{\text{id}}$ is the identity.

Proposition 6.24. *There exists a pseudofunctor $\tilde{\varepsilon}: P\tilde{K}\mathcal{C} \rightarrow \mathcal{C}$ defined by*

$$\begin{aligned} [\vec{m}, \{\vec{x}, \vec{c}\}] &\mapsto \tilde{\varepsilon}_{\vec{m}}(\{\vec{x}, \vec{c}\}), \\ [\bar{\phi}, \{\vec{f}, \vec{g}\}] &\mapsto \tilde{\varepsilon}_{\vec{n}}(\{\vec{f}, \vec{g}\}) \circ \tilde{\varepsilon}_{\bar{\phi}}, \\ [\bar{\phi}, \vec{a}] &\mapsto \tilde{\varepsilon}_{\vec{n}}(\vec{a}) * 1_{\tilde{\varepsilon}_{\bar{\phi}}}. \end{aligned}$$

Proof. It is easy to check that $\tilde{\varepsilon}_{\text{id}} = \text{id}$. Moreover, for composable morphisms in \mathcal{A} ,

$$\vec{m} \xrightarrow{\bar{\phi}} \vec{n} \xrightarrow{\bar{\psi}} \vec{p}$$

we have the following equality by Remark 6.10.

$$\begin{array}{ccc} \begin{array}{c} A\tilde{K}\mathcal{C}(\underline{m}_+) \\ \downarrow \bar{\phi}_* \\ A\tilde{K}\mathcal{C}(\underline{n}_+) \\ \downarrow \bar{\psi}_* \\ A\tilde{K}\mathcal{C}(\underline{p}_+) \end{array} & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} & \begin{array}{c} A\tilde{K}\mathcal{C}(\underline{m}_+) \\ \downarrow (\bar{\psi}\bar{\phi})_* \\ A\tilde{K}\mathcal{C}(\underline{p}_+) \end{array} \\ & \begin{array}{c} \tilde{\varepsilon}_{\bar{\phi}} \\ \parallel \\ \tilde{\varepsilon}_{\bar{n}} \\ \parallel \\ \tilde{\varepsilon}_{\bar{p}} \end{array} & \begin{array}{c} \tilde{\varepsilon}_{\bar{m}} \\ \parallel \\ \tilde{\varepsilon}_{\bar{\psi}\bar{\phi}} \\ \parallel \\ \tilde{\varepsilon}_{\bar{p}} \end{array} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

By the bicategorical Grothendieck construction of [CCG11, §3.2], $\tilde{\varepsilon}$ is a pseudofunctor. Since $\tilde{\varepsilon}_{\text{id}}$ is the identity pseudo natural transformation and $\tilde{\varepsilon}_m$ preserves identity 1-cells, so does $\tilde{\varepsilon}$. \square

Proposition 6.25. *Let \mathcal{C} be a permutative Gray-monoid. The pseudofunctor $\tilde{\varepsilon}$ is the underlying functor of a symmetric monoidal pseudofunctor*

$$P\tilde{K}\mathcal{C} \rightarrow \mathcal{C}.$$

Proof. Following Definition 3.3, we need two transformations, one relating the unit in $P\tilde{K}\mathcal{C}$ with the unit in \mathcal{C} and another one relating the monoidal product in $P\tilde{K}\mathcal{C}$ with the one in \mathcal{C} . The unit in the monoidal structure of $P\tilde{K}\mathcal{C}$ is the pair $(((), *))$. By definition we have $\tilde{\varepsilon}(((), *)) = e$ and by Remark 6.23 $\tilde{\varepsilon}$ strictly preserves the identity 1-cell of e . Thus the

unit is preserved strictly and we can take the first transformation to be the identity. We now define the second transformation

$$\begin{array}{ccc}
 P\tilde{K}\mathcal{C} \times P\tilde{K}\mathcal{C} & \xrightarrow{\tilde{\varepsilon} \times \tilde{\varepsilon}} & \mathcal{C} \times \mathcal{C} \\
 \square \downarrow & \swarrow \chi & \downarrow \oplus \\
 P\tilde{K}\mathcal{C} & \xrightarrow{\tilde{\varepsilon}} & \mathcal{C}
 \end{array}$$

which can be taken to have identity components on objects, as both composites give the same value when evaluated at a pair of objects. Thus χ will be a pseudonatural transformation with identity components, or in other words an invertible icon [Lac10]. We must now construct components for 1-cells, and check two axioms, one for the units and one for composition. These diagrams appear in, e.g., [Lac10]. The component at a pair of 1-morphisms

$$\begin{aligned}
 [\bar{\phi}, \{\vec{f}, \vec{\gamma}\}] &: [\vec{m}, \{\vec{x}, \vec{c}\}] \rightarrow [\vec{n}, \{\vec{y}, \vec{d}\}] \\
 [\bar{\psi}, \{\vec{f}', \vec{\gamma}'\}] &: [\vec{m}', \{\vec{x}', \vec{c}'\}] \rightarrow [\vec{n}', \{\vec{y}', \vec{d}'\}]
 \end{aligned}$$

is given by the pasting below. Recall that $\tilde{\varepsilon}_{\bar{\phi}}(\vec{x}, \vec{c})$ is defined in Display (6.19) as a composite 1-cell $\beta \circ \oplus_i c_{\bar{\phi}}^i$. For ease of notation we let

$$c_{\bar{\phi}} = \oplus_i c_{\bar{\phi}}^i$$

and let

$$\begin{array}{c}
 c_{\bar{\phi}} \tilde{\varepsilon}_{\vec{m}}(\vec{x}, \vec{c}) = \bigoplus_i \bigoplus_{j \in p(\bar{\phi}, i)} x_{\bar{\phi}^{-1}(\underline{n}_j)}^i. \\
 \\
 \begin{array}{ccccc}
 & \tilde{\varepsilon}_{\vec{n}}(\vec{y}, \vec{d}) \oplus \tilde{\varepsilon}_{\vec{m}'}(\vec{x}', \vec{c}') & & & \\
 & \nearrow \tilde{\varepsilon}_{\vec{n}}(f) \oplus \text{id} & \searrow \text{id} \oplus c_{\bar{\psi}} & & \\
 & \tilde{\varepsilon}_{\vec{n}} \bar{\phi}_* \{\vec{x}, \vec{c}\} \oplus \tilde{\varepsilon}_{\vec{m}'} \{\vec{x}', \vec{c}'\} & & & \\
 & \beta \oplus \text{id} & \text{id} \oplus c_{\bar{\psi}} & & \\
 & \uparrow \Sigma^{-1} & & & \\
 & \tilde{\varepsilon}_{\vec{n}} \bar{\phi}_* \{\vec{x}, \vec{c}\} \oplus \tilde{\varepsilon}_{\vec{m}'} \{\vec{x}', \vec{c}'\} & & & \\
 & \nearrow c_{\bar{\phi}} \tilde{\varepsilon}_{\vec{m}}(\vec{x}, \vec{c}) \oplus \tilde{\varepsilon}_{\vec{m}'}(\vec{x}', \vec{c}') & \text{id} \oplus c_{\bar{\psi}} & & \\
 & & \tilde{\varepsilon}_{\vec{n}} \bar{\phi}_* \{\vec{x}, \vec{c}\} \oplus c_{\bar{\psi}} \tilde{\varepsilon}_p(\vec{x}', \vec{c}') & & \\
 & & \beta \oplus \text{id} & & \\
 & & \uparrow \Sigma^{-1} & & \\
 & & \tilde{\varepsilon}_{\vec{n}} \bar{\phi}_* \{\vec{x}, \vec{c}\} \oplus \tilde{\varepsilon}_{\vec{n}'} \bar{\psi}_* \{\vec{x}', \vec{c}'\} & & \\
 & & \nearrow c_{\bar{\phi}} \oplus \text{id} & \text{id} \oplus \beta & \\
 & & & \tilde{\varepsilon}_{\vec{n}}(\vec{y}, \vec{d}) \oplus \tilde{\varepsilon}_{\vec{n}'} \bar{\psi}_* \{\vec{x}', \vec{c}'\} & \\
 & & & \beta \oplus \text{id} & \text{id} \oplus \tilde{\varepsilon}_{\vec{n}'}(f') \\
 & & & \uparrow \Sigma^{-1} & \\
 & & & \tilde{\varepsilon}_{\vec{n}}(\vec{f}) \oplus \text{id} & \\
 & & & \nearrow \tilde{\varepsilon}_{\vec{n}}(\vec{f}) \oplus \tilde{\varepsilon}_{\vec{n}'}(f') & \\
 & & & & \tilde{\varepsilon}_{\vec{n}}(\vec{f}) \oplus c_{\bar{\psi}} \\
 & & & & \nearrow c_{\bar{\phi}} \oplus c_{\bar{\psi}} \\
 & & & & \tilde{\varepsilon}_{\vec{n}}(\vec{f}) \oplus \tilde{\varepsilon}_{\vec{n}'}(f')
 \end{array}
 \end{array}$$

If either f or ψ is the identity, then the single instance of Σ above is also the identity, which immediately implies the unit axiom for χ . The composition axiom involves a much larger diagram that we omit, noting that it follows from the definitions of the cells involved and the Gray tensor product axioms.

There are then three invertible modifications to construct in order to show that $\tilde{\varepsilon}$ has a monoidal structure (see, e.g., [McC00, App. A]). Two concern the unit object, and one easily checks that these can be chosen to be the identity, and the third concerns

using the monoidal product on a triple of objects. This last one, usually denoted ω , can also be taken to be the identity using the Gray tensor product axioms. Then there are two axioms to check, but each consists of only identity 2-cells, thus they both trivially commute.

There is one further modification, U , needed for a symmetric monoidal structure on $\tilde{\varepsilon}$. One can check that $\tilde{\varepsilon}$ sends the 1-cell β in $P\tilde{K}\mathcal{C}$ to the 1-cell β in \mathcal{C} , and then the 2-naturality of β (see Definition 3.27) can be used to show that we can take U to be the identity as well. Below we will check the final three symmetric monoidal pseudofunctor axioms which all concern the interaction between U , the monoidal structure on $\tilde{\varepsilon}$, and the modifications $R_{-|-}, R_{--|}, v$ in the source and target. Diagrams for these are drawn in [McC00, App. A,B].

For the first axiom, one pasting diagram consists of two instances of ω which are both the identity, a naturality 2-cell for β which is the identity since one of the components is an identity 1-cell, one instance of U which is the identity, and one final 2-cell. This 2-cell is obtained by pasting together instances of the pseudofunctoriality isomorphisms for $\tilde{\varepsilon}$. Recalling the expression for the braiding in Display (4.30), these 2-cells are all special cases of the pseudofunctoriality isomorphisms in which the 1-cell has the form $[\bar{\sigma}, \{\text{id}, \text{id}\}]$ for a block permutation σ . Combining Remark 6.23 with the formulas for $\tilde{\varepsilon}$ in Proposition 6.24, these are all identity 2-cells. A similar argument, with the additional observation that $\tilde{\varepsilon}$ strictly preserves identity 1-cells (see Display (6.20)), shows that the other pasting diagram for this axiom also consists of only identity 2-cells. Thus the axiom reduces to the statement that the identity 2-cell is equal to itself.

The second axiom, relating $R_{-|-}$ to U , is analogous. The third axiom relates U to the syllepsis $v: \beta^2 \cong 1$, and follows by the same line of reasoning, only this time using that the pseudofunctoriality isomorphism

$$\tilde{\varepsilon}(\beta) \circ \tilde{\varepsilon}(\beta) \cong \tilde{\varepsilon}(\beta \circ \beta)$$

is the identity. □

Combining the formulas of Propositions 6.9 and 6.24 and Remark 5.20, one sees immediately that $\tilde{\varepsilon}$ is natural with respect to strict functors of permutative Gray-monoids.

Proposition 6.26. *The functor $\tilde{\varepsilon} = \tilde{\varepsilon}_{\mathcal{C}}: P\tilde{K}\mathcal{C} \rightarrow \mathcal{C}$ is natural in \mathcal{C} with respect to strict functors of permutative Gray-monoids.*

Remark 6.27. For general pseudofunctors (F, θ) , $\tilde{\varepsilon}$ is not strictly natural. We do expect that $\tilde{\varepsilon}$ satisfies a weak naturality, induced by the pseudofunctoriality transformation θ , but have not pursued those details.

We now describe the homotopy-theoretic properties of $\tilde{\varepsilon}$. We use these to show in Theorem 6.44 that the homotopy theory of permutative Gray-monoids is equivalent to that of permutative 2-categories.

Proposition 6.28. *For a permutative Gray-monoid \mathcal{C} , $\tilde{\varepsilon}: P\tilde{K}\mathcal{C} \rightarrow \mathcal{C}$ is a weak equivalence of 2-categories.*

Proof. By Proposition 6.12, $\tilde{K}\mathcal{C}$ is a special Γ -2-category and $\tilde{K}\mathcal{C}(\underline{1}_+) \cong \mathcal{C}$. Hence by Lemma 5.22 the first map in the composite below is a weak equivalence

$$\mathcal{C} \cong \tilde{K}\mathcal{C}(\underline{1}_+) \rightarrow P\tilde{K}\mathcal{C} \xrightarrow{\tilde{\varepsilon}} \mathcal{C}.$$

Since the composite is identity on \mathcal{C} , the result follows. □

Remark 6.29. Proposition 6.28 demonstrates a genuinely new phenomenon for symmetric monoidal structures appearing at the 2-dimensional level. Namely, we have a strict

notion of symmetric monoidal structure which models all weak homotopy types of a more general symmetric monoidal structure but does not model all categorical equivalence types.

Indeed, there exist permutative Gray-monoids which are not equivalent (in the categorical sense) to any permutative 2-category. The fundamental reason for this is that Gray-monoids model (unstable) connected 3-types and, as is well-known, strict monoidal 2-categories cannot. The 2-cells Σ are necessary to model the generally nontrivial Whitehead product

$$\pi_2 \times \pi_2 \rightarrow \pi_3.$$

Simpson [Sim12, §2.7] shows, for example, that the 3-type of S^2 cannot be modeled by any strict 3-groupoid and therefore certainly not by any strict monoidal 2-groupoid.

The example of [SP11, Example 2.30] is a permutative Gray-monoid for which, for the same reason, cannot be equivalent to any strict monoidal 2-category. In particular, it cannot be equivalent to a permutative 2-category.

We would like to use Proposition 6.28 to show that every permutative Gray-monoid is weakly equivalent, in the category of permutative Gray-monoids and strict maps, to a permutative 2-category. However $\tilde{\varepsilon}$ does not achieve this directly since it is not a strict symmetric monoidal map. We do achieve a zigzag of weak equivalences in Proposition 6.30, however, by applying our variant of the strictification of [SP11] given in Theorem 3.14.

Proposition 6.30. *Applying the construction in Theorem 3.14 to $\tilde{\varepsilon}$ yields a natural zigzag of strict symmetric monoidal weak equivalences between $P\tilde{K}\mathcal{C}$ and \mathcal{C} . Hence these are naturally isomorphic in $\text{ho}\text{PermGrayMon}$.*

Proof. The second part of Theorem 3.14, when applied to $\tilde{\varepsilon}$, yields the square

$$\begin{array}{ccc} P\tilde{K}\mathcal{C}^{qst} & \xrightarrow{\nu} & P\tilde{K}\mathcal{C} \\ \tilde{\varepsilon}^{qst} \downarrow & \simeq & \downarrow \tilde{\varepsilon} \\ \mathcal{C}^{qst} & \xrightarrow{\nu} & \mathcal{C} \end{array}$$

which only commutes up to a symmetric monoidal equivalence as indicated. All of the objects in this square are permutative Gray-monoids, and the only map which is not strict is $\tilde{\varepsilon}$. Both instances of ν are symmetric monoidal biequivalences, and the symmetric monoidal equivalence 2-cell filling the square yields a homotopy upon taking nerves, so the 2-out-of-3 property for weak equivalences shows that $\tilde{\varepsilon}^{qst}$ is a weak equivalence as $\tilde{\varepsilon}$ is by Proposition 6.28. Thus our zigzag of strict symmetric monoidal weak equivalences is

$$P\tilde{K}\mathcal{C} \xleftarrow{\nu} P\tilde{K}\mathcal{C}^{qst} \xrightarrow{\tilde{\varepsilon}^{qst}} \mathcal{C}^{qst} \xrightarrow{\nu} \mathcal{C}.$$

Given a strict map of permutative Gray-monoids $F: \mathcal{B} \rightarrow \mathcal{C}$, the zigzag above is natural in F since both ν and $\tilde{\varepsilon}$ are. \square

6.3. K -theory for permutative 2-categories. Recall that $\text{Perm2Cat}_{\text{nop}}$ denotes the category of permutative 2-categories and normal oplax functors between them. Using the additional rigidity of permutative 2-categories we give a second, somewhat simpler, K -theory functor

$$K: \text{Perm2Cat}_{\text{nop}} \rightarrow \Gamma\text{-}2\text{Cat}.$$

It is this version of K -theory that we will use to construct our equivalences of homotopy theories in Section 7.

Throughout this section, let $(\mathcal{C}, \oplus, e, \beta)$ be a permutative 2-category.

Construction 6.31. For a finite pointed set $\underline{n}_+ \in \mathcal{F}$, $K\mathcal{C}(\underline{n}_+)$ is a 2-category defined as follows:

- i. The objects of $K\mathcal{C}(\underline{n}_+)$ are the same as those of $\tilde{K}\mathcal{C}(\underline{n}_+)$.
- ii. A 1-morphism from $\{x_s, c_{s,t}\}$ to $\{x'_s, c'_{s,t}\}$ is a system of 1-morphisms $\{f_s\}$, with $f_s: x_s \rightarrow x'_s$, such that $f_\emptyset = \text{id}_e$ and for all s and t , the diagram

$$\begin{array}{ccc} x_{s \cup t} & \xrightarrow{c_{s,t}} & x_s \oplus x_t \\ f_{s \cup t} \downarrow & & \downarrow f_s \oplus f_t \\ x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x'_t. \end{array}$$

commutes.

- iii. A 2-morphism

$$\begin{array}{ccc} \{x_s, c_s\} & \xrightarrow{\{f_s\}} & \{x'_s, c'_s\} \\ & \Downarrow \{\alpha_s\} & \\ & \xrightarrow{\{g_s\}} & \end{array}$$

is a collection of 2-morphisms $\alpha_s: f_s \rightarrow g_s$ such that $\alpha_\emptyset = \text{id}_{\text{id}_e}$ and for all s and t , the equality

$$\begin{array}{ccc} x_{s \cup t} & \xrightarrow{c_{s,t}} & x_s \oplus x_t \\ g_{s \cup t} \leftarrow \begin{array}{c} \alpha_{s \cup t} \\ \hphantom{\alpha_{s \cup t}} \swarrow \\ \alpha_s \end{array} & f_{s \cup t} & \downarrow f_s \oplus f_t \\ x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x'_t \end{array} = \begin{array}{ccc} x_{s \cup t} & \xrightarrow{c_{s,t}} & x_s \oplus x_t \\ g_{s \cup t} \downarrow & & \\ x'_{s \cup t} & \xrightarrow{c'_{s,t}} & x'_s \oplus x'_t \\ & \leftarrow \begin{array}{c} \alpha_s \oplus \alpha_t \\ \hphantom{\alpha_s \oplus \alpha_t} \swarrow \\ \alpha_s \end{array} & f_{s \cup t} \end{array}$$

holds.

Composition of 1-morphisms and 2-morphisms is done componentwise. We emphasize that this definition is possible only because $\oplus: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a 2-functor. It is easy to check that $K\mathcal{C}(\underline{n}_+)$ is a 2-category. Given $\phi: \underline{m}_+ \rightarrow \underline{n}_+$ in \mathcal{F} , the 2-functor $\phi_*: K\mathcal{C}(\underline{m}_+) \rightarrow K\mathcal{C}(\underline{n}_+)$ is defined in an analogous way to the one for \tilde{K} (Construction 6.8).

Remark 6.32. This construction is a *Cat*-enrichment of the standard *K*-theory for permutative categories initiated by Segal [Seg74]. See, e.g., [Man10].

Definition 6.33. Let \mathcal{C}, \mathcal{D} be a pair of permutative 2-categories. A strict functor of permutative 2-categories $F: \mathcal{C} \rightarrow \mathcal{D}$ is a *stable equivalence* if KF is a stable equivalence of Γ -2-categories. We let $(\text{Perm2Cat}, \mathcal{S})$ denote the relative category of permutative 2-categories and stable equivalences.

We have the analogue of Proposition 6.12 for K .

Proposition 6.34. *The Γ -2-category $K\mathcal{C}$ is special, with $K\mathcal{C}(\underline{1}_+)$ isomorphic to \mathcal{C} .*

Since \mathcal{C} is a permutative 2-category it is also a permutative Gray-monoid, so we also have the weaker construction $\tilde{K}\mathcal{C}$ of Section 6.1. As the construction $K\mathcal{C}$ is a specialization of the former construction, we have an inclusion

$$K\mathcal{C} \rightarrow \tilde{K}\mathcal{C}$$

given by

$$\begin{aligned} \{x_s, c_{s,t}\} &\mapsto \{x_s, c_{s,t}\}, \\ \{f_s\} &\mapsto \{f_s, \text{id}\}, \\ \{\alpha_s\} &\mapsto \{\alpha_s, \text{id}\}. \end{aligned}$$

Proposition 6.35. *For a permutative 2-category \mathcal{C} , the inclusion*

$$K\mathcal{C} \rightarrow \tilde{K}\mathcal{C}$$

is a strict map of Γ -2-categories and a levelwise weak equivalence. The inclusion is natural with respect to strict functors of permutative 2-categories, i.e., strict symmetric monoidal 2-functors.

Proof. The first statement is obvious, and therefore we get a commutative square

$$\begin{array}{ccc} K\mathcal{C}(\underline{n}_+) & \longrightarrow & \tilde{K}\mathcal{C}(\underline{n}_+) \\ \downarrow & & \downarrow \\ \mathcal{C}^n & \longrightarrow & \mathcal{C}^n \end{array}$$

using the projection maps, Proposition 6.34, and the fact that the map $K\mathcal{C}(\underline{1}_+) \rightarrow \tilde{K}\mathcal{C}(\underline{1}_+)$ is the identity. Both vertical maps and the bottom map are weak equivalences, so the top is as well, proving the second statement. Naturality with respect to strict functors of permutative 2-categories is then just a straightforward check using the definitions. \square

Remark 6.36. If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a normal oplax map of permutative 2-categories, then the map of Γ -2-categories $\tilde{K}F$ constructed in Proposition 6.9 in fact defines a map

$$KF: K\mathcal{C} \rightarrow K\mathcal{D}.$$

We have the following as an immediate consequence of Propositions 6.13 and 6.35, together with the fact that stable equivalences satisfy the 2 out of 3 property.

Proposition 6.37. *The inclusion $\text{Perm2Cat} \hookrightarrow \text{PermGrayMon}$ preserves and reflects stable equivalences, so defines a relative functor $(\text{Perm2Cat}, \mathcal{S}) \rightarrow (\text{PermGrayMon}, \mathcal{S})$.*

We also have a further result about K .

Proposition 6.38. *The functor $K: \text{Perm2Cat} \rightarrow \Gamma\text{-2Cat}$ sends weak equivalences of permutative 2-categories to levelwise weak equivalences of Γ -2-categories, and therefore is a relative functor $(\text{Perm2Cat}, \mathcal{W}) \rightarrow (\Gamma\text{-2Cat}, \mathcal{W})$.*

Remark 6.39. Note that the version of the previous proposition in which \mathcal{W} is replaced by \mathcal{S} is in fact merely the definition of stable equivalences in Perm2Cat .

The additional strictness of $K\mathcal{C}$ for a permutative 2-category \mathcal{C} provides a stricter version of the map $\tilde{\varepsilon}$ constructed above.

Definition 6.40. For a permutative 2-category \mathcal{C} , let $\varepsilon: PK\mathcal{C} \rightarrow \mathcal{C}$ be the composite

$$PK\mathcal{C} \rightarrow P\tilde{K}\mathcal{C} \xrightarrow{\tilde{\varepsilon}} \mathcal{C}.$$

For $\bar{\phi}: \vec{m} \rightarrow \vec{n}$ in \mathcal{A} , we let $\varepsilon_{\vec{m}}$ and $\varepsilon_{\bar{\phi}}$ denote the corresponding composites with $AK\mathcal{C} \rightarrow A\tilde{K}\mathcal{C}$.

Although $\tilde{\varepsilon}$ is only a weak symmetric monoidal functor of permutative Gray-monoids, the composite ε enjoys a number of stricter properties.

Proposition 6.41. *Let \mathcal{C} be a permutative 2-category and let $\bar{\phi}: \vec{m} \rightarrow \vec{n}$ in \mathcal{A} . Then*

- i. $\varepsilon_{\vec{m}}$ is a 2-functor.
- ii. $\varepsilon_{\vec{\phi}}$ is 2-natural.
- iii. ε is a strict symmetric monoidal functor.

Proof. The first claim holds because the composite in Definition 6.16 is 2-functorial when \oplus is a 2-functor. The second holds because the 2-cells in Display (6.20) are identities when \mathcal{C} is a permutative 2-category. The third holds upon noting that χ in the proof of Proposition 6.25 is the identity 2-cell when \mathcal{C} is a permutative 2-category. \square

Taking a *Cat*-enrichment of [Man10, Thm 4.7] or by specializing Proposition 6.26 we have the following statement about naturality of ε .

Proposition 6.42. *Let \mathcal{C} be a permutative 2-category. Then*

$$\varepsilon: PK\mathcal{C} \rightarrow \mathcal{C}$$

is natural with respect to strict maps of permutative 2-categories, and lax natural with respect to lax maps.

Proposition 6.43. *For a permutative 2-category \mathcal{C} , $\varepsilon: PK\mathcal{C} \rightarrow \mathcal{C}$ is a weak equivalence.*

Proof. The map ε is the composite of two maps by definition. The first is a weak equivalence since it is P applied to the levelwise equivalence of Proposition 6.35. The second is a weak equivalence by Proposition 6.28. \square

We are finally able to prove our first important equivalence of homotopy theories.

Theorem 6.44. *The inclusion $Perm2Cat \hookrightarrow PermGrayMon$ induces an equivalence of homotopy theories*

$$(Perm2Cat, \mathcal{W}) \xrightarrow{\sim} (PermGrayMon, \mathcal{W}).$$

Proof. First note that the inclusion obviously preserves weak equivalences. In the other direction, we have the functor $P\tilde{K}: PermGrayMon \rightarrow Perm2Cat$. The functor \tilde{K} preserves weak equivalences by Proposition 6.13, and the functor P preserves weak equivalences by Proposition 5.23, so $P\tilde{K}$ defines a relative functor. One natural zigzag of weak equivalences is given in Proposition 6.30 and the other is

$$P\tilde{K}\mathcal{C} \leftarrow PK\mathcal{C} \rightarrow \mathcal{C}.$$

Thus Corollary 2.9 shows that we have an equivalence of homotopy theories. \square

Corollary 6.45. *The functors*

$$Perm2Cat \hookrightarrow PermGrayMon,$$

$$PermGrayMon \xrightarrow{P\tilde{K}} Perm2Cat$$

establish an equivalence of homotopy categories

$$\text{ho } PermGrayMon \simeq \text{ho } Perm2Cat.$$

7. EQUIVALENCES OF HOMOTOPY THEORIES

This section establishes our main result, that there are equivalences of homotopy theories between permutative Gray-monoids, permutative 2-categories, and Γ -2-categories when each is equipped with the relative category structure given by stable equivalences. We begin by constructing a lax unit η to complement the strict counit ε of Definition 6.40. These provide a lax adjunction in the following sense: In Sections 7.2 and 7.3 we show that the unit has components which are Γ -lax maps

$$X \rightarrow KPX$$

and satisfies an oplax naturality condition with respect to Γ -lax maps.

The key technical result is that η furnishes us with a natural zigzag of stable equivalences between the identity and KP . In Sections 7.5 and 7.6 we prove that the composites $K\varepsilon \circ \eta$ and $\varepsilon \circ P\eta$ are identities on the objects of Γ - $2Cat$ and $Perm2Cat$, respectively. These triangle identities, together with some simple naturality statements, show that η is a stable equivalence (Theorem 7.23). The final step is then to convert this lax stable equivalence into a zigzag (in this case, a span) of strict stable equivalences using the methods of Section 4.5.

7.1. Construction of the unit. Throughout this section let X be a Γ -2-category. We define a Γ -lax map

$$\eta: X \rightarrow KPX.$$

Notation 7.1. For $s \subset \underline{m}$, define

$$\pi^s: \underline{m}_+ \rightarrow |\underline{s}|_+, \quad s = \{i_1, \dots, i_{|s|}\}$$

$$\pi^s(i) = \begin{cases} 0, & \text{if } i \notin s \\ k, & \text{if } i = i_k \in s. \end{cases}$$

Note. The map π^s is the unique surjective, order-preserving map $\underline{m}_+ \rightarrow |\underline{s}|_+$ which maps the complement of s to the basepoint.

Notation 7.2. For $s, t \subset \underline{m}$ with $s \cap t = \emptyset$, define a map in \mathcal{A}

$$\pi^{s,t}: |s \cup t| \rightarrow (|s|, |t|)$$

that corresponds to the partition of the ordered set $s \cup t$ into the disjoint union of s and t .

Remark 7.3. For $x \in X(\underline{m}_+)$, we have $\pi_*^s x \in AX(|s|) = X(|s|_+)$ and

$$\pi_*^{s,t} \pi_*^{s \cup t} x = \pi_*^s x \times \pi_*^t x$$

in $AX(|s|, |t|) = X(|s|_+) \times X(|t|_+)$.

Now for a finite set \underline{m}_+ we define $\eta = \eta_{X, \underline{m}_+}$ using the notation of Remark 6.7.

Definition 7.4. For a 0-cell $x \in X(\underline{m}_+)$,

$$(7.5) \quad \eta(x) = \left\{ \begin{array}{lcl} s & \mapsto & [|s|, \pi_*^s x] \\ s, t & \mapsto & [\pi^{s,t}, \text{id}_{\pi_*^s x \times \pi_*^t x}] \end{array} \right\}$$

By Remark 7.3, this gives a well-defined 0-cell of $KPX(\underline{m}_+)$.

For a 1-cell $x \xrightarrow{f} y$,

$$\eta(f) = \{s \mapsto [\text{id}_{|s|}, \pi_*^s f]\}.$$

For a 2-cell $\alpha: f \rightarrow g$,

$$\eta(\alpha) = \{s \mapsto [\text{id}_{|s|}, \pi_*^s \alpha]\}.$$

One easily verifies that $\eta(f)$ and $\eta(\alpha)$ satisfy the structure diagrams of 1- and 2-cells of $KPX(\underline{m}_+)$ by checking each $s, t \subset \underline{m}$ with $s \cap t = \emptyset$. It is also easy to check that $\eta_{X, \underline{m}_+}$ is a 2-functor.

7.2. Laxity of the unit components. For $\phi: \underline{m}_+ \rightarrow \underline{n}_+$ and $s \subset \underline{n}$ we have

$$\begin{aligned} (\eta(\phi_* x))_s &= [|s|, \pi_*^s \phi_* x] \in PX \\ (\phi_* \eta(x))_s &= [|\phi^{-1}(s)|, \pi_*^{\phi^{-1}(s)} x] \in PX. \end{aligned}$$

These are not equal, so $\eta = \eta_X$ is not a strict map of Γ -2-categories. Instead, η_X naturally has the structure of a Γ -lax map (Definition 4.2), as we now describe.

Notation 7.6. For $\phi: \underline{m}_+ \rightarrow \underline{n}_+$ in \mathcal{F} and $s \subset \underline{n}$, the map ϕ^s is defined by reindexing $\phi|_{\phi^{-1}(s)}$ so that the following diagram of pointed sets commutes.

$$\begin{array}{ccc} \underline{m}_+ & \xrightarrow{\phi} & \underline{n}_+ \\ \pi^{\phi^{-1}(s)} \downarrow & & \downarrow \pi^s \\ |\phi^{-1}(s)|_+ & \xrightarrow{\phi^s_+} & |s|_+ \end{array}$$

Definition 7.7. Let X be a Γ -2-category. For $\phi: \underline{m}_+ \rightarrow \underline{n}_+$ in \mathcal{F} and $x \in X(\underline{m}_+)$, let

$$\eta_\phi(x): \phi_* \eta(x) \rightarrow \eta(\phi_* x)$$

be given by

$$\eta_\phi(x) = \{s \mapsto [\phi^s, \text{id}_{\pi^s \phi_* x}]\}.$$

Each $[\phi^s, \text{id}_{\pi^s \phi_* x}]$ is a well-defined morphism in PX since

$$(7.8) \quad \phi_*^s \pi_*^{\phi^{-1}(s)} x = \pi_*^s \phi_* x.$$

These 1-cells are natural in x and hence we have the following.

Proposition 7.9. Let X be a Γ -2-category. The 1-cells $\eta_\phi = \eta_{X, \phi}$ give η_X the structure of a Γ -lax map $X \rightarrow KPX$.

7.3. Naturality of the unit. In this section we discuss the oplax naturality of the Γ -lax maps η_X for varying X . This is sufficient to imply that η induces a natural transformation in the homotopy category of Γ -2-categories and Γ -lax maps (Proposition 7.21) and we use this to show η is a stable equivalence (Theorem 7.23). We observe, furthermore, that η is strictly natural with respect to strict Γ maps (Corollary 7.14) and this is essential to the proof of Theorem 7.27.

Let $h: X \rightarrow Y$ be a Γ -lax map. The diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & KPX \\ h \downarrow & & \downarrow KPh \\ Y & \xrightarrow{\eta_Y} & KPY \end{array}$$

does not generally commute. We define a Γ -transformation

$$\lambda: \eta h \rightarrow KPh \eta$$

as follows. For readability, we omit subscripts X and Y on η . For each $\underline{m}_+ \in \mathcal{F}$ and each $x \in X(\underline{m}_+)$, we combine Definition 7.4 and Remarks 5.20 and 6.36 to compute

$$(7.10) \quad (\eta h)_{\underline{m}_+}(x) = \left\{ \begin{array}{l l} (s \subset \underline{m}) & \mapsto [|s|, \pi_*^s h(x)] \\ \left(\begin{array}{l} s, t \subset \underline{m} \\ s \cap t = \emptyset \end{array} \right) & \mapsto [\pi^{s, t}, \text{id}] \end{array} \right\}$$

and

$$(7.11) \quad (KPh\eta)_{\underline{m}_+}(x) = \left\{ \begin{array}{lcl} (s \subset \underline{m}) & \mapsto & [s, h(\pi^s_* x)] \\ \left(\begin{array}{c} s, t \subset \underline{m} \\ s \cap t = \emptyset \end{array} \right) & \mapsto & [\pi^{s,t}, h_{\pi^{s,t}}] \end{array} \right\}.$$

Definition 7.12. For each $\underline{m}_+ \in \mathcal{F}$ and each $x \in X(\underline{m}_+)$, we define component 1-cells

$$\lambda_{\underline{m}_+}(x): (\eta h)_{\underline{m}_+}(x) \rightarrow (KPh\eta)_{\underline{m}_+}(x)$$

in $KPY(\underline{m}_+)$. These are given by

$$\lambda_{\underline{m}_+}(x) = \{s \mapsto [\text{id}_{|s|}, h_{\pi^s}(x)]\}.$$

Checking that this is a legitimate 1-cell reduces to verifying $\pi^s_* h_{\pi^{s \cup t}} = h_{\pi^s \pi^{s \cup t}} = h_{\pi^s}$. This follows because h is a Γ -lax map.

In Lemma 7.16 we prove that these are the components of a 2-natural transformation

$$\lambda_{\underline{m}_+}: \eta h_{\underline{m}_+} \rightarrow KPh\eta_{\underline{m}_+}$$

for each $\underline{m}_+ \in \mathcal{F}$. In Lemma 7.17 we verify the single axiom for a Γ -transformation (see Definition 4.5). Together these prove the following.

Proposition 7.13. *For a Γ -lax map $h: X \rightarrow Y$, λ gives a Γ -transformation*

$$\lambda: \eta h \rightarrow KPh\eta.$$

Corollary 7.14. *For a strict Γ -map $h: X \rightarrow Y$, the Γ -transformation λ is the identity. Thus η is strictly natural on strict maps, i.e., $\eta h = KPh\eta$ if h is a strict Γ -map.*

Remark 7.15. We do expect that η is a lax transformation when $\Gamma\text{-2Cat}_l$ is regarded as a 2-category (see Remark 4.4) and P, K are extended to this structure. However we have not needed to pursue those details.

Lemma 7.16. *Each $\lambda_{\underline{m}_+}$ is 2-natural.*

Proof. It suffices to check 2-naturality for each subset $s \subset \underline{m}$. Given $s \subset \underline{m}$, the 1-cell $[\text{id}_{|s|}, h_{\pi^s}]$ is a 2-natural transformation from $[|s|, \pi^s h(-)]$ to $[|s|, h\pi^s(-)]$ because h is Γ -lax and hence h_{π^s} is 2-natural. \square

Lemma 7.17. *For all $\phi: \underline{m}_+ \rightarrow \underline{n}_+$ the following cube commutes:*

$$\begin{array}{ccccc}
& X(\underline{n}_+) & \xrightarrow{\eta_{\underline{n}_+}} & KPX(\underline{n}_+) & \\
X(\phi) \nearrow & \Downarrow \eta_\phi & \nearrow KPX(\phi) & \searrow KPh_{\underline{n}_+} & \\
X(\underline{m}_+) & \xrightarrow{\eta_{\underline{m}_+}} & KPX(\underline{m}_+) & \parallel & KPY(\underline{n}_+) \\
h_{\underline{m}_+} \searrow & \Downarrow \lambda_{\underline{m}_+} & \nearrow KPh_{\underline{m}_+} & \nearrow KPY(\phi) & \\
& Y(\underline{m}_+) & \xrightarrow{\eta_{\underline{m}_+}} & KPY(\underline{m}_+) & \\
& & \parallel & & \\
& X(\underline{n}_+) & \xrightarrow{\eta_{\underline{n}_+}} & KPX(\underline{n}_+) & \\
X(\phi) \nearrow & \searrow h_{\underline{n}_+} & \Downarrow \lambda_{\underline{n}_+} & \searrow KPh_{\underline{n}_+} & \\
X(\underline{m}_+) & \xrightarrow{h_\phi \uparrow\!\!\!\uparrow} & Y(\underline{n}_+) & \xrightarrow{\eta_{\underline{n}_+}} & KPY(\underline{n}_+) \\
h_{\underline{m}_+} \searrow & \nearrow Y(\phi) & \Downarrow \eta_\phi & \nearrow KPY(\phi) & \\
& Y(\underline{m}_+) & \xrightarrow{\eta_{\underline{m}_+}} & KPY(\underline{m}_+) &
\end{array}$$

Proof. Commutativity of the cube in the statement of the lemma reduces to checking, for each object $x \in X(\underline{m}_+)$, commutativity of the following rectangle in $KPY(\underline{n}_+)$.

$$\begin{array}{ccccc}
\phi_* \eta h(x) & \xrightarrow{\eta_\phi} & \eta \phi_* h(x) & \xrightarrow{\eta(h_\phi)} & \eta h(\phi_* x) \\
\phi_* \lambda_{\underline{m}_+} \downarrow & & & & \downarrow \lambda_{\underline{n}_+} \\
\phi_* KPh \eta(x) & \xrightarrow{\text{id}} & KPh \phi_* \eta(x) & \xrightarrow{KPh(\eta_\phi)} & KPh \eta(\phi_* x)
\end{array}$$

Formulas for the objects in this diagram are given as in Displays (7.10) and (7.11). Formulas for the 1-cells are given similarly, using Definition 7.7 in addition. Commutativity of the rectangle above then reduces to checking, for each $s \subset \underline{n}$, commutativity of the following rectangle in PY .

$$\begin{array}{ccccc}
[|\phi^{-1}(s)|, \pi_*^{\phi^{-1}(s)} h x] & \xrightarrow{[\phi^s, \text{id}]} & [|s|, \pi_*^s \phi_* h x] & \xrightarrow{[\text{id}, \pi_*^s(h_\phi)]} & [|s|, \pi_*^s h \phi_* x] \\
(7.18) \quad [\text{id}, h_{\pi_*^{\phi^{-1}(s)}}] \downarrow & & & & \downarrow [\text{id}, h_{\pi^s}] \\
[|\phi^{-1}(s)|, h \pi_*^{\phi^{-1}(s)} x] & \xrightarrow{\text{id}} & [|s|, h \pi_*^{\phi^{-1}(s)} x] & \xrightarrow{[\phi^s, h_{\phi^s}]} & [|s|, h \pi_*^s \phi_* x]
\end{array}$$

The top right composite of Display (7.18) is $[\phi^s, h_{\pi^s} \pi_*^s(h_\phi)]$ by Remark 5.19. Likewise, the bottom left composite is $[\phi^s, h_{\phi^s} \circ (\phi^s)_*(h_{\pi_*^{\phi^{-1}(s)}})]$. To verify that the second components are equal, consider Display (7.19) and Display (7.20) respectively: both are special cases of the axioms for Γ -lax maps (see Definition 4.2).

(7.19)

$$\begin{array}{ccccc}
X(\underline{m}_+) & \xrightarrow{\phi_*} & X(\underline{n}_+) & \xrightarrow{\pi^s_*} & X(|s|_+) \\
h_{\underline{m}_+} \downarrow & \swarrow h_\phi & h_{\underline{n}_+} \downarrow & \swarrow h_{\pi^s} & h_{|s|_+} \downarrow \\
Y(\underline{m}_+) & \xrightarrow{\phi_*} & Y(\underline{n}_+) & \xrightarrow{\pi^s_*} & Y(|s|_+)
\end{array} = \begin{array}{ccccc}
X(\underline{m}_+) & \xrightarrow{\pi^s_* \phi_*} & X(|s|_+) & & \\
h_{\underline{m}_+} \downarrow & \swarrow h_{\pi^s \phi} & h_{|s|_+} \downarrow & & \\
Y(\underline{m}_+) & \xrightarrow{\pi^s_* \phi_*} & Y(|s|_+) & &
\end{array}$$

(7.20)

$$\begin{array}{ccccc}
X(\underline{m}_+) & \xrightarrow{\pi_*^{\phi^{-1}(s)}} & X(|\phi^{-1}(s)|_+) & \xrightarrow{\phi_*^s} & X(|s|_+) \\
h_{\underline{m}_+} \downarrow & \swarrow h_{\pi^{\phi^{-1}(s)}} & h_{|\phi^{-1}(s)|_+} \downarrow & \swarrow h_{\phi^s} & h_{|s|_+} \downarrow \\
Y(\underline{m}_+) & \xrightarrow{\pi_*^{\phi^{-1}(s)}} & Y(|\phi^{-1}(s)|_+) & \xrightarrow{\phi_*^s} & Y(|s|_+)
\end{array} = \begin{array}{ccccc}
X(\underline{m}_+) & \xrightarrow{\phi_*^s \pi_*^{\phi^{-1}(s)}} & X(|s|_+) & & \\
h_{\underline{m}_+} \downarrow & \swarrow h_{\phi^s \pi^{\phi^{-1}(s)}} & h_{|s|_+} \downarrow & & \\
Y(\underline{m}_+) & \xrightarrow{\phi_*^s \pi_*^{\phi^{-1}(s)}} & Y(|s|_+) & &
\end{array}$$

By definition of ϕ^s we have $\phi^s \pi^{\phi^{-1}(s)} = \pi^s \phi$ (see Display (7.8)). Hence the right hand sides of Display (7.19) and Display (7.20) are equal. \square

7.4. Two equivalences. Two lengthy calculations aside, we are now ready to state and prove our main theorems. We show that the unit η is a stable equivalence, and use this to prove our two main equivalences of homotopy theories. We begin with the following, which is immediate from Corollary 4.15 and Proposition 7.13.

Proposition 7.21. *In the homotopy category $\text{ho}\Gamma\text{-2Cat}_l$, the unit $\eta: \text{Id} \Rightarrow KP$ is a natural transformation of functors.*

Our next lemma is a key step in the proof that η is a stable equivalence, and demonstrates the importance of the triangle identities. This lemma depends on two careful computations of the composites $(K\varepsilon_{PX}) \circ (KP\eta_X)$ and $(K\varepsilon_{PX}) \circ (\eta_{KPX})$. As the computations are completely self-contained, we state their key application in Lemma 7.22 and postpone the explicit details to Propositions 7.28 and 7.29 in Sections 7.5 and 7.6, respectively.

Lemma 7.22. *Let X be a Γ -2-category. In the homotopy category $\text{ho}\Gamma\text{-2Cat}_l$ we have $\eta_{KPX} = KP\eta_X$.*

Proof. First, note that ε_{PX} is a weak equivalence by Proposition 6.43. So the result follows from the two equalities

$$(K\varepsilon_{PX}) \circ (KP\eta_X) = \text{id}_{KPX} = (K\varepsilon_{PX}) \circ (\eta_{KPX}).$$

The first of these follows from Proposition 7.29 and functoriality of K with respect to strict symmetric monoidal functors (see Proposition 6.9). The second equality is proved in Proposition 7.28. \square

Theorem 7.23. *Let X be a Γ -2-category. The unit $\eta = \eta_X$ is a stable equivalence.*

Proof. We verify directly that for any very special Γ -2-category, Z , the induced map

$$\text{ho}\Gamma\text{-2Cat}_l(KPX, Z) \xrightarrow{\eta^*} \text{ho}\Gamma\text{-2Cat}_l(X, Z)$$

is a bijection of sets. To do this, we explicitly construct an inverse as follows. First, given and object \underline{m}_+ in \mathcal{F} , Consider the diagram of 2-categories

$$\begin{array}{ccc} Z(\underline{m}_+) & \xrightarrow{\eta_{Z,m}} & KPZ(\underline{m}_+) \\ \downarrow & & \downarrow \\ Z(\underline{1}_+)^m & \xrightarrow{\eta_{Z,1}^m} & KPZ(\underline{1}_+)^m \end{array}$$

where the vertical maps are the Segal maps. This diagram commutes up to a 2-natural transformation. Since Z and KPZ are special, the two vertical maps are weak equivalences. Note that $\eta_{Z,1}$ is precisely the map of Lemma 5.22, thus it is a weak equivalence. It follows then that η_Z is a levelwise weak equivalence. For $h: X \rightarrow Z$, let $F(h)$ be given by the zigzag below:

$$KPX \xrightarrow{KPh} KPZ \xleftarrow{\eta_Z} Z.$$

To show that F defines an inverse bijection of sets, we must verify $\eta^*(F(h)) = h$ for $h: X \rightarrow Z$ and $F(\eta^*(k)) = k$ for $k: KPX \rightarrow Z$. These follow by verifying that the diagrams below commute in $\text{ho}\Gamma\text{-2Cat}_l$.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & KPX \\ h \downarrow & & \downarrow KPh \\ Z & \xrightarrow{\eta_Z} & KPZ \end{array} \quad \begin{array}{ccc} KPX & \xrightarrow{k} & Z \\ \downarrow KP(\eta_X) & & \downarrow \eta_Z \\ KPKPX & \xrightarrow{KPh} & KPZ \end{array}$$

The first commutes by Proposition 7.21, and the second by Lemma 7.22 and Proposition 7.21. \square

We can now use that η is a stable equivalence to verify that P preserves stable equivalences as well as weak equivalences.

Proposition 7.24. *The functor $P: \Gamma\text{-2Cat}_l \rightarrow \text{Perm2Cat}$ preserves and reflects stable equivalences, and so is a relative functor $(\Gamma\text{-2Cat}_l, \mathcal{S}) \rightarrow (\text{Perm2Cat}, \mathcal{S})$.*

Proof. Let $h: X \rightarrow Y$ be a Γ -lax map between Γ -2-categories. Then by Proposition 7.21, we have

$$[\eta][h] = [KPh][\eta]$$

in $\text{ho}\Gamma\text{-2Cat}_l$, so this equation holds in $\text{Ho}\Gamma\text{-2Cat}_l$ as well. By Theorem 7.23, we know that η is a stable equivalence, and since these satisfy the 2-out-of-3 property, h is a stable equivalence if and only if KPh is. But by definition (see Definition 6.11) KPh is a stable equivalence if and only if Ph is, and therefore h is a stable equivalence if and only if Ph is. \square

Theorem 7.25. *The inclusion $\text{Perm2Cat} \hookrightarrow \text{PermGrayMon}$ induces an equivalence of homotopy theories*

$$(\text{Perm2Cat}, \mathcal{S}) \xrightarrow{\simeq} (\text{PermGrayMon}, \mathcal{S}).$$

Proof. The inclusion $\text{Perm2Cat} \hookrightarrow \text{PermGrayMon}$ preserves stable equivalences by Proposition 6.37. The rest of the proof is now the same as that of Theorem 6.44, using Proposition 7.24. \square

We now turn to our final equivalence of homotopy theories.

Lemma 7.26. *The Γ -lax maps $\eta_X: X \rightarrow KPX$ induce a natural zigzag of stable equivalences from X to KPX in the category Γ -2Cat.*

Proof. Since η is natural with respect to strict maps by Corollary 7.14, it defines a functor

$$\eta: \Gamma\text{-}2Cat \rightarrow \Gamma\text{-}2Cat_{\text{lax, str}}^{\bullet \rightarrow \bullet}.$$

Composing this with the functor E of Corollary 4.47 yields a functor

$$E\eta_{(-)}: \Gamma\text{-}2Cat \rightarrow \text{Span}(\Gamma\text{-}2Cat).$$

This means that given a strict Γ -map $h: X \rightarrow Y$, one has the following commuting diagram in $\Gamma\text{-}2Cat$.

$$\begin{array}{ccccc} X & \xleftarrow{\omega} & E\eta_X & \xrightarrow{\nu} & KPX \\ h \downarrow & & E\eta_h \downarrow & & \downarrow KPh \\ Y & \xleftarrow{\omega} & E\eta_Y & \xrightarrow{\nu} & KPY \end{array}$$

Therefore the functor $E\eta$ provides, for each Γ -2-category X , a zigzag of strict Γ -maps which is natural with respect to strict Γ -maps.

The maps ω are levelwise weak equivalences by Proposition 4.36 and hence stable equivalences. Further, commutativity of the lefthand square shows that $X \rightarrow E\eta_X$ preserves stable equivalences, and so is a relative functor. Corollary 4.49 implies that the maps ν are stable equivalences because, by Theorem 7.23, the components of η are stable equivalences. \square

Theorem 7.27. *The functors*

$$K: \text{Perm2Cat} \rightarrow \Gamma\text{-}2Cat,$$

$$P: \Gamma\text{-}2Cat \rightarrow \text{Perm2Cat}$$

induce an equivalence of homotopy theories between $(\text{Perm2Cat}, \mathcal{S})$ and $(\Gamma\text{-}2Cat, \mathcal{S})$.

Proof. By Corollary 2.9, we need to construct a natural zigzag of stable equivalences between the composites KP, PK and the respective identity functors. The first of those is given in Lemma 7.26, while the second is given by the transformation ε using Proposition 6.42 and Proposition 6.43. \square

7.5. **The composite $K\varepsilon_{\mathcal{C}} \circ \eta_{K\mathcal{C}}$.** In this section we prove the following:

Proposition 7.28. *Let \mathcal{C} be a permutative 2-category. Then the composite Γ -lax map of Γ -2-categories is the identity on $K\mathcal{C}$:*

$$K\mathcal{C} \xrightarrow{\eta_{K\mathcal{C}}} KPK\mathcal{C} \xrightarrow{K\varepsilon_{\mathcal{C}}} K\mathcal{C}.$$

Proof. Throughout this section we work at a fixed permutative 2-category \mathcal{C} and will often simply write $\eta = \eta_{K\mathcal{C}}$ for simplicity.

We first describe

$$\eta_{K\mathcal{C}}: K\mathcal{C} \rightarrow KPK\mathcal{C}.$$

For $\underline{m}_+ \in \mathcal{F}$ we use Definition 7.4 to describe $\eta = \eta_{\underline{m}_+}$ on the following 0-, 1-, and 2-cells in $K\mathcal{C}(\underline{m}_+)$:

$$\begin{array}{ccc} \{x, c\} & \begin{array}{c} \xrightarrow{\{f\}} \\ \Downarrow \{\alpha\} \\ \xrightarrow{\{g\}} \end{array} & \{y, d\} \end{array}$$

On 0-cells:

$$\eta(\{x, c\}) = \left\{ \begin{array}{l} (s \subset \underline{m}) \rightarrow [s, \pi_*^s\{x, c\}] \\ \left(\begin{array}{l} s, t \subset \underline{m} \\ s \cap t = \emptyset \end{array} \right) \rightarrow [\pi^{s,t}, \text{id}_{\pi_*^s\{x, c\} \times \pi_*^t\{x, c\}}] \end{array} \right\}$$

where, unraveling the definition of π^s , we have

$$\pi_*^s\{x, c\} = \left\{ \begin{array}{l} (r \subset |s|) \rightarrow x_{(\pi^s)^{-1}(r)} \\ \left(\begin{array}{l} r, u \subset |s| \\ r \cap u = \emptyset \end{array} \right) \rightarrow c_{(\pi^s)^{-1}(r), (\pi^s)^{-1}(u)} \end{array} \right\}.$$

Note that we make use of Remark 7.3 to see that $\eta(\{x, c\})_{s,t}$ is a well-defined map in $PK\mathcal{C}$:

$$[\pi^{s,t}, \text{id}_{\pi_*^s\{x, c\} \times \pi_*^t\{x, c\}}] : [|s \cup t|, \pi_*^{s \cup t}\{x, c\}] \rightarrow [(|s|, |t|), \pi_*^s\{x, c\} \times \pi_*^t\{x, c\}].$$

On 1-cells:

$$\eta(\{f\}) = \{(s \subset \underline{m}) \rightarrow [\text{id}_{|s|}, \pi_*^s\{f\}]\}.$$

On 2-cells:

$$\eta(\{\alpha\}) = \{(s \subset \underline{m}) \rightarrow [\text{id}_{|s|}, \pi_*^s\{\alpha\}]\}.$$

Now we describe the composite $K\epsilon_{\mathcal{C}} \circ \eta_{K\mathcal{C}}$ using the description of $\eta_{K\mathcal{C}}$ above, the definition of ϵ (described explicitly for $\bar{\epsilon}$ in Proposition 6.24) and the functoriality of K described in Proposition 6.9.

On 0-cells:

$$K\epsilon \circ \eta(\{x, c\}) = \left\{ \begin{array}{l} (s \subset \underline{m}) \rightarrow \epsilon_{|s|} [s, \pi_*^s\{x, c\}] = x_{(\pi^s)^{-1}(\underline{s})} = x_s \\ \left(\begin{array}{l} s, t \subset \underline{m} \\ s \cap t = \emptyset \end{array} \right) \rightarrow \epsilon_{(|s|, |t|)} (\text{id}_{\pi_*^s\{x, c\} \times \pi_*^t\{x, c\}}) \circ \epsilon_{\pi^{s,t}} \end{array} \right\}.$$

Now $\epsilon_{\vec{m}}(\text{id}_{\vec{x}}) = \text{id}_{\epsilon_{\vec{m}}(\vec{x})}$ and, unraveling the definition of $\epsilon_{\vec{\phi}}$ for $\vec{\phi} = \pi^{s,t}$, one finds

$$\epsilon_{\pi^{s,t}} = c_{s,t} : x_{s \cup t} \rightarrow x_s \oplus x_t.$$

On 1-cells:

$$K\epsilon \circ \eta(\{f\}) = \{s \mapsto \epsilon_{|s|} [\text{id}_{|s|}, \pi_*^s\{f\}]\} = \{f\}.$$

Similarly, on 2-cells:

$$K\epsilon \circ \eta(\{\alpha\}) = \{\alpha\}.$$

Therefore $K\epsilon \circ \eta = (K\epsilon \circ \eta_{K\mathcal{C}})_{\underline{m}_+}$ is the identity on 0-, 1-, and 2-cells of $K\mathcal{C}(\underline{m}_+)$.

Now ϵ is a strict map of 2-categories by Proposition 6.41 and therefore $K\epsilon$ is a strict map of Γ -2-categories. However, η is only a Γ -lax map and so for a map of finite sets $\phi : \underline{m}_+ \rightarrow \underline{n}_+$, we have the following diagram.

$$\begin{array}{ccccc} K\mathcal{C}(\underline{m}_+) & \xrightarrow{\eta} & KPK\mathcal{C}(\underline{m}_+) & \xrightarrow{K\epsilon} & K\mathcal{C}(\underline{m}_+) \\ \phi \downarrow & \swarrow \eta_\phi & \phi \downarrow & = & \phi \downarrow \\ K\mathcal{C}(\underline{n}_+) & \xrightarrow{\eta} & KPK\mathcal{C}(\underline{n}_+) & \xrightarrow{K\epsilon} & K\mathcal{C}(\underline{n}_+) \end{array}$$

To verify that $K\epsilon \circ \eta$ is the identity as a lax transformation, we need to verify that this 2-cell composite – formed by whiskering η_ϕ with $K\epsilon$ – is the identity. To do this, we must consider

$$K\epsilon(\eta_\phi) : K\epsilon(\phi_* \eta(\{x, c\})) \rightarrow K\epsilon(\eta(\phi_* \{x, c\})).$$

First, we describe the source and target of η_ϕ more explicitly:

$$\phi_* \eta(\{x, c\}) = \left\{ \begin{array}{lcl} (s \subset \underline{n}) & \mapsto & \left[|\phi^{-1}(s)|, \pi_*^{\phi^{-1}(s)}\{x, c\} \right] \\ \left(\begin{array}{c} s, t \subset \underline{n} \\ s \cap t = \emptyset \end{array} \right) & \mapsto & \left[\pi^{\phi^{-1}(s), \phi^{-1}(t)}, \text{id}_{\pi_*^{\phi^{-1}(s)}\{x, c\} \times \pi_*^{\phi^{-1}(t)}\{x, c\}} \right] \end{array} \right\}$$

and

$$\eta(\phi_* \{x, c\}) = \left\{ \begin{array}{lcl} (s \subset \underline{n}) & \mapsto & \left[|s|, \pi_*^s \phi_* \{x, c\} \right] \\ \left(\begin{array}{c} s, t \subset \underline{n} \\ s \cap t = \emptyset \end{array} \right) & \mapsto & \left[\pi^{s, t}, \text{id}_{\pi_*^s \phi_* \{x, c\} \times \pi_*^t \phi_* \{x, c\}} \right] \end{array} \right\}.$$

Note that

$$\phi_* \{x, c\} = \left\{ \begin{array}{lcl} s & \mapsto & x_{\phi^{-1}(s)} \\ s, t & \mapsto & c_{\phi^{-1}(s), \phi^{-1}(t)} \end{array} \right\}.$$

Using the notation of Section 7.2, the map η_ϕ is given by

$$\{(s \subset \underline{n}) \mapsto [\phi^s, \text{id}_{\pi_*^s \phi_* \{x, c\}}]\}.$$

Hence we have

$$K\varepsilon(\eta_\phi) = \{(s \subset \underline{n}) \mapsto \varepsilon([\phi^s, \text{id}_{\pi_*^s \phi_* \{x, c\}}]) = \varepsilon(\text{id}_{\pi_*^s \phi_* \{x, c\}}) \circ \varepsilon_{\phi^s}\}.$$

Unraveling the definitions, one finds

$$\varepsilon(\text{id}_{\pi_*^s \phi_* \{x, c\}}) = \text{id}_{x_{\phi^{-1}(s)}}$$

and by Remark 6.22

$$\varepsilon_{\phi^s} = \text{id}_{x_{\phi^{-1}(s)}}.$$

Therefore $K\varepsilon(\eta_\phi) = \{s \mapsto \text{id}_{x_{\phi^{-1}(s)}}\} = \text{id}_{\{x, c\}}$ and this completes the proof. \square

7.6. The composite $\varepsilon_{PX} \circ P\eta_X$. In this section we prove the following:

Proposition 7.29. *Let X be a Γ -2-category. Then the composite*

$$PX \xrightarrow{P\eta_X} PKPX \xrightarrow{\varepsilon_{PKPX}} PX$$

of symmetric monoidal functors is the identity on PX .

Proof. First, recall that ε_{PX} is a strict symmetric monoidal functor by Proposition 6.41 and $P\eta_X = \mathcal{A} \int A\eta_X$ is a strict symmetric monoidal functor by Propositions 4.31 and 5.17.

To show that $\varepsilon \circ P\eta = \text{id}$ as maps of permutative 2-categories, we check this equality on 0-, 1-, and 2-cells. We use the definition of η in Definition 7.4, the description of P in Section 5.3, and the definition of ε (see Definition 6.40 and Proposition 6.24). On 0-cells, the definitions of ε and $A\eta$ immediately give:

$$\begin{aligned} \varepsilon \circ P\eta([\vec{m}, \vec{x}]) &= \varepsilon([\vec{m}, A\eta_{\vec{m}}(\vec{x})]) \\ &= \varepsilon_{\vec{m}}(A\eta_{\vec{m}}(\vec{x})) \\ &= [\vec{m}, \vec{x}]. \end{aligned}$$

On 1-cells:

$$\begin{aligned} \varepsilon \circ P\eta([\vec{\phi}, \vec{f}]) &= \varepsilon([\vec{\phi}, A\eta_{\vec{n}}(\vec{f}) \circ A\eta_{\vec{\phi}}]) \\ &= \varepsilon_{\vec{n}}(A\eta_{\vec{n}}(\vec{f}) \circ A\eta_{\vec{\phi}}) \circ \varepsilon_{\vec{\phi}}. \end{aligned}$$

Because, by Remark 5.19, each 1-cell is given as a composite

$$[\bar{\phi}, \vec{f}] = [\text{id}, \vec{f}] [\bar{\phi}, \text{id}],$$

and because both $P\eta$ and ε are 2-functors (see Propositions 4.31, 5.17 and 6.41), it suffices to consider 1-cells of the form $[\text{id}, \vec{f}]$ or $[\bar{\phi}, \text{id}]$. In the first case, it is immediate that $\varepsilon \circ P\eta([\text{id}, \vec{f}]) = [\text{id}, \vec{f}]$ because $A\eta_{\text{id}} = \text{id}$ and $\varepsilon_{\text{id}} = \text{id}$. In the second case, we have

$$\varepsilon \circ P\eta([\bar{\phi}, \text{id}]) = \varepsilon_{\vec{n}}(A\eta_{\bar{\phi}}) \circ \varepsilon_{\bar{\phi}}.$$

This composite is, by definition, the component at an object $\vec{x} \in X(\vec{m})$ of the transformation formed by the pasting below. As in Remark 5.20 and Proposition 6.24, the object \vec{x} has been suppressed from the notation.

$$\begin{array}{ccccc}
 AX(\vec{m}) & \xrightarrow{A\eta_{\vec{m}}} & AKPX(\vec{m}) & & \\
 \Pi_i \phi_i \downarrow & \swarrow \Pi_i A\eta_{\phi_i} & \downarrow \Pi_i \phi_i & \swarrow \oplus_i \varepsilon_{\phi_i} & \curvearrowright \varepsilon_{\vec{m}} \\
 \Pi_i \prod_{j \in \mathbb{P}} X(\underline{n_j}_+) & \xrightarrow{\Pi_i \prod_{j \in \mathbb{P}} \eta_{\underline{n_j}_+}} & \Pi_i \prod_{j \in \mathbb{P}} KPX(\underline{n_j}_+) & \xrightarrow{\oplus_i \varepsilon_{(n_j)_j \in \mathbb{P}}} & PX \\
 \tau \downarrow & \parallel & \tau \downarrow & \swarrow \beta & \nearrow \varepsilon_{\vec{n}} \\
 AX(\vec{n}) & \xrightarrow{A\eta_{\vec{n}}} & AY(\vec{n}) & &
 \end{array} \tag{7.30}$$

The lower 2-cell β is given, as in the right-hand side of Display (6.19), by the 1-cell β for the relevant permutation of terms

$$\oplus_i \varepsilon_{(n_j)_j \in \mathbb{P}}((\phi_i)_* \{x^i, c^i\}) \xrightarrow{\beta} \varepsilon_{\vec{n}}(\bar{\phi}_* \{\vec{x}, \vec{c}\}).$$

This is precisely the permutation of components in Remark 5.11 which makes the following diagram of finite sets commute.

$$\begin{array}{ccc}
 \coprod_i \underline{m_i} & \xrightarrow{\bar{\phi}} & \coprod_j \underline{n_j} \\
 \searrow \Pi_i \phi_i & & \swarrow \coprod_i \coprod_{j \in \mathbb{P}} \underline{n_j}
 \end{array}$$

Therefore, restricting to the top half of Display (7.30), it suffices to show that for each i we have

$$(7.31) \quad \varepsilon_{(n_j)_j \in \mathbb{P}}(\eta_{\phi_i}) \circ \varepsilon_{\phi_i} = [\phi_i, \text{id}].$$

This left-hand composite is, for each $x^i \in X(\underline{m_i}_+)$, given as follows:

$$(7.32) \quad \text{ev}_{\underline{m_i}} \eta_{\underline{m_i}_+}(x^i) \xrightarrow{\varepsilon_{\phi_i}} \left(\prod_{j \in \mathbb{P}} \text{ev}_{\underline{n_j}} \right) \phi_{i*} \eta_{\underline{m_i}_+}(x^i) \xrightarrow{\left(\prod_{j \in \mathbb{P}} \text{ev}_{\underline{n_j}} \right) (A\eta_{\phi_i})} \left(\prod_{j \in \mathbb{P}} \text{ev}_{\underline{n_j}} \eta_{\underline{n_j}_+} \right) \phi_{i*}(x^i).$$

To understand this composite, we make critical use of Notations 5.8, 6.17 and 7.6. From the definition of η in Display (7.5), we have

$$\eta_{\underline{m}_i}(x^i) = \left\{ \begin{array}{l} (s \subset \underline{m}_i) \rightarrow [s, \pi_*^s x^i] \\ (s, t \subset \underline{m}_i) \rightarrow [\pi^{s,t}, \text{id}_{\pi_*^s x^i \times \pi_*^t x^i}] \\ s \cap t = \emptyset \end{array} \right\}.$$

By definition (see Display (6.19)), ε_{ϕ_i} is the map

$$[|m_i|, x^i] \rightarrow \left[(|\phi_i^{-1}(n_j)|)_{j \in \mathbb{P}}, \prod_{j \in \mathbb{P}} \pi_*^{\phi_i^{-1}(n_j)} x^i \right]$$

induced by the partition of \underline{m}_i into $\coprod_{j \in \mathbb{P}} |\phi_i^{-1}(n_j)|$ and the identity on $\prod_{j \in \mathbb{P}} \pi_*^{\phi_i^{-1}(n_j)} x^i$. Next, from the definition of $A\eta_{\phi_i}$ in Display (5.15) we have

$$A\eta_{\phi_i} = \prod_{j \in \mathbb{P}} \left\{ (s \subset \underline{n}_j) \rightarrow [(\phi_{i,j})^s, \text{id}] \right\}$$

and hence

$$(7.33) \quad \left(\prod_{j \in \mathbb{P}} \text{ev}_{\underline{n}_j} \right) (A\eta_{\phi_i}) = \prod_{j \in \mathbb{P}} [(\phi_{i,j})^{\underline{n}_j}, \text{id}].$$

Each $(\phi_{i,j})^{\underline{n}_j}$ is given by the restriction of $\phi_{i,j}$ to $(\phi_{i,j})^{-1}(\underline{n}_j) \subset \underline{m}_{i+}$ and a reindexing $|\phi_{i,j})^{-1}(\underline{n}_j)| \cong (\phi_{i,j})^{-1}(\underline{n}_j)$. Hence the product in Display (7.33) is given in the first component by the disjoint union of the maps

$$\phi_{i,j}|_{(\phi_{i,j})^{-1}(\underline{n}_j)} = \phi_i|_{(\phi_{i,j})^{-1}(\underline{n}_j)} = \phi_i|_{(\phi_i)^{-1}(\underline{n}_j)}$$

appropriately reindexed to have source $|\phi_i)^{-1}(\underline{n}_j)|$.

The composite (7.32) of maps in PX is given by composing the first components in \mathcal{A} , since both are identities in their second component. This is

$$\underline{m}_i \longrightarrow \coprod_{j \in \mathbb{P}(\bar{\phi}, i)} |\phi_i^{-1}(\underline{n}_j)| \cong \coprod_{j \in \mathbb{P}(\bar{\phi}, i)} \phi_i^{-1}(\underline{n}_j) \longrightarrow \coprod_{j \in \mathbb{P}(\bar{\phi}, i)} \underline{n}_j.$$

But this is precisely the decomposition of ϕ_i described in Remark 5.11 and is equal to ϕ_i . Therefore the composite (7.31) is equal to $[\phi_i, \text{id}]$ as claimed.

Lastly, on 2-cells we have

$$\begin{aligned} \varepsilon \circ P\eta([\bar{\phi}, \vec{\alpha}]) &= \varepsilon([\bar{\phi}, A\eta_{\bar{n}}(\vec{\alpha}) * 1_{A\eta_{\bar{\phi}}})] \\ &= \varepsilon_{\bar{n}}(A\eta_{\bar{n}}(\vec{\alpha}) * 1_{A\eta_{\bar{\phi}}}) * 1_{\varepsilon_{\bar{\phi}}} \\ &= [\bar{\phi}, \vec{\alpha}] \end{aligned}$$

where the last equality follows by again considering the cases $\vec{\alpha} = \text{id}$ and $\bar{\phi} = \text{id}$ separately. This completes the proof. \square

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